Integral Equalities for Functions of Unbounded Spectral Operators in Banach Spaces

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ABSTRACT. The work is dedicated to investigating a limiting procedure for extending "local" integral operator equalities to the "global " ones in the sense explained below, and to applying it to obtaining generalizations of the Newton-Leibnitz formula for operator-valued functions for a wide class of unbounded operators. The integral equalities considered in the paper have the following form

$$g(R_F)\int f_x(R_F)\,d\,\mu(x) = h(R_F). \tag{1}$$

They involve functions of the kind

$$X \ni x \mapsto f_x(R_F) \in B(F),$$

where X is a general locally compact space, F runs in a suitable class of Banach subspaces of a fixed complex Banach space G, in particular $F = G$. The integrals are with respect to a general complex Radon measure on X and with respect to the $\sigma(B(F), \mathcal{N}_F)-$ topology on $B(F)$, where $\mathcal{N}_F$ is a suitable subset of $B(F)^*$, the topological dual of $B(F)$. R_F is a possibly unbounded scalar type spectral operator in F such that $\sigma(R_F) \subseteq \sigma(R_G)$, and for all $x \in X$ f_x and g, h are complex-valued Borelian maps on the spectrum $\sigma(R_G)$ of R_G. If $F \neq G$ we call the integral equality (1) "local", while if $F = G$ we call it "global".

Contents

Introduction vii

Summary of the main results xv

Chapter 1. Extension theorem. The case of the strong operator topology 1
1. Key lemma 1
2. Extension theorem for strong operator integral equalities 8
3. Generalization of the Newton-Leibnitz formula 15
4. Application to resolvents of unbounded scalar type spectral operators in a Banach space G 24

Chapter 2. Extension theorem. The case of the topology $\sigma(B(G), \mathcal{N})$ 29
1. Introduction 29
2. Existence of the weak-integral with respect to the $\sigma(B(G), \mathcal{N})-$ topology 29
3. Commutation and restriction properties 35
4. Extension theorem for integral equalities with respect to the $\sigma(B(G), \mathcal{N})-$ topology 43
5. Generalization of the Newton-Leibnitz formula 48
Acknowledgments 50

Bibliography 51

Introduction

The work is dedicated to investigating a limiting procedure for extending "local" integral operator equalities to the "global" ones in the sense explained below, and to applying it to obtaining generalizations of the Newton-Leibnitz formula for operator-valued functions for a wide class of unbounded operators.

The integral equalities considered in the work have the following form

$$g(R_F)\int f_x(R_F)\,d\,\mu(x)=h(R_F). \tag{2}$$

They involve functions of the kind

$$X\ni x\mapsto f_x(R_F)\in B(F),$$

where X is a general locally compact space, F is a suitable Banach subspace of a fixed complex Banach space G, for example $F=G$. The integrals are with respect to a general complex Radon measure on X and with respect to the $\sigma(B(F),\mathcal{N}_F)-$ topology [1] on $B(F)$. R_F is a possibly unbounded scalar type spectral operator in F such that $\sigma(R_F)\subseteq\sigma(R_G)$, and for all $x\in X$, f_x and g,h are complex-valued Borelian maps on the spectrum $\sigma(R_G)$ of R_G.

If $F\neq G$ we call the integral equalities (2) "local", while if $F=G$ we call them "global".

Let G be a complex Banach space and $B(G)$ the Banach algebra of all bounded linear operators on G. *Scalar type spectral operators* in G were defined in [**DS**] Definition $18.2.12.$ [2] (see Section 1), and were created for providing a general Banach space with a class of unbounded linear operators for which it is possible to establish a Borel functional calculus similar to the well-known one for unbounded self-adjoint operators in a Hilbert space.

We start with the following useful formula [3] for the resolvent of T

$$(T-\lambda\mathbf{1})^{-1}=i\int_{-\infty}^{0}e^{-it\lambda}e^{itT}\,d\,t. \tag{3}$$

[1] Here $\mathcal{N}_F$ is a suitable subset of $B(F)^*$, the topological dual of $B(F)$, associated with the resolution of the identity of R_F.

[2] For the special case of bounded spectral operators on G see [**Dow**].

[3] An important application of this formula is made in proving the well-known Stone theorem for strongly continuous semigroups of unitary operators in Hilbert space, see Theorem $12.6.1.$ of [**DS**]. In [**Dav**] it has been used for showing the equivalence of uniform convergence in strong operator topology of a one-parameter semigroup depending on a parameter and the convergence in strong operator topology of the resolvents of the corresponding generators, Theorem 3.17.

Here $\lambda \in \mathbb{C}$ is such that $Im(\lambda) > 0$ and the integral is with respect to the Lebesgue measure and with respect to the strong operator topology on $B(G)$ [4]. It is known that this formula holds for

(1) any bounded operator $T \in B(G)$ on a complex Banach space G with real spectrum $\sigma(T)$, see for example [**LN**];
(2) any infinitesimal generator T of a strongly continuous semi-group in a Banach space, see Corollary $8.1.16.$ of [**DS**], in particular for any unbounded self-adjoint operator $T : \mathbf{D}(T) \subset H \to H$ in a complex Hilbert space H.

Next we consider a more general case. Let S be an entire function and $L > 0$, then the Newton-Leibnitz formula

$$R \int_{u_1}^{u_2} \frac{d\,S}{d\,\lambda}(tR)\,d\,t = S(u_2 R) - S(u_1 R), \tag{4}$$

for all $u_1, u_2 \in [-L, L]$ was known for any element R in a Banach algebra $\mathcal{A}$, where $S(tR)$ and $\frac{dS}{d\lambda}(tR)$ are understood in the standard framework of analytic functional calculus on Banach algebras, while the integral is with respect to the Lebesgue measure in the norm topology on $\mathcal{A}$ see for example [**Rud, Dieu, Schw**]. If E is the resolution of the identity of R then for all $U \in \mathcal{B}(\mathbb{C})$ [5]

$$\mathfrak{L}_E^\infty(U) \doteqdot \left\{ f : \mathbb{C} \to \mathbb{C} \mid \|f\chi_U\|_\infty^E < \infty \right\}.$$

Here $\chi_U : \mathbb{C} \to \mathbb{C}$ is equal to 1 in U and 0 in $\complement U$ and for all maps $F : \mathbb{C} \to \mathbb{C}$

$$\|F\|_\infty^E \doteqdot E - ess \sup_{\lambda \in \mathbb{C}} |F(\lambda)| \doteqdot \inf_{\{\delta \in \mathcal{B}(\mathbb{C}) \mid E(\delta) = \mathbf{1}\}} \sup_{\lambda \in \delta} |F(\lambda)|.$$

See [**DS**].

We say (see Definition 2.11) that $\mathcal{N}$ is an $E-$**appropriate set** if

(1) $\mathcal{N} \subseteq B(G)^*$ linear subspace;
(2) $\mathcal{N}$ separates the points of $B(G)$, namely

$$(\forall T \in B(G) - \{\mathbf{0}\})(\exists\,\omega \in \mathcal{N})(\omega(T) \neq 0);$$

(3) $(\forall \omega \in \mathcal{N})(\forall \sigma \in \mathcal{B}(\mathbb{C}))$ we have

$$\omega \circ \mathcal{R}(E(\sigma)) \in \mathcal{N} \text{ and } \omega \circ \mathcal{L}(E(\sigma)) \in \mathcal{N}. \tag{5}$$

Moreover, we say that $\mathcal{N}$ is an $E-$**appropriate set with the duality property** if in addition

$$\mathcal{N}^* \subseteq B(G). \tag{6}$$

Here for any Banach algebra $\mathcal{A}$, so in particular for $\mathcal{A} = B(G)$, we set $\mathcal{R} : \mathcal{A} \to \mathcal{A}^{\mathcal{A}}$ and $\mathcal{L} : \mathcal{A} \to \mathcal{A}^{\mathcal{A}}$ defined by

$$\begin{cases} \mathcal{R}(T) : \mathcal{A} \ni h \mapsto Th \in \mathcal{A} \\ \mathcal{L}(T) : \mathcal{A} \ni h \mapsto hT \in \mathcal{A}, \end{cases} \tag{7}$$

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[4] Notice that if $\zeta \doteqdot -i\lambda$ and $Q \doteqdot iT$, then the equality (3) turns into

$$(Q + \zeta\mathbf{1})^{-1} = \int_0^\infty e^{-t\zeta} e^{-Qt}\,d\,t,$$

which is referred in $IX.1.3.$ of [**Kat**] as the fact that the resolvent of Q is the *Laplace* trasform of the semigroup e^{-Qt}. Applications of this formula to perturbation theory are in $IX.2.$ of [**Kat**].

[5] $\mathcal{B}(\mathbb{C})$ is the class of all Borelian sets of $\mathbb{C}$.

for all $T \in \mathcal{A}$. Notice that for all $T, h \in \mathcal{A}$ we have $\|\mathcal{R}(T)(h)\|_{\mathcal{A}} \leq \|T\|_{\mathcal{A}}\|h\|_{\mathcal{A}}$, and $\|\mathcal{L}(T)(h)\|_{\mathcal{A}} \leq \|T\|_{\mathcal{A}}\|h\|_{\mathcal{A}}$, so

$$\mathcal{R}(T), \mathcal{L}(T) \in B(\mathcal{A}) \tag{8}$$

with

$$\|\mathcal{R}(T)\|_{B(\mathcal{A})} \leq \|T\|_{\mathcal{A}}, \|\mathcal{L}(T)\|_{B(\mathcal{A})} \leq \|T\|_{\mathcal{A}}. \tag{9}$$

Since $\mathcal{L}$ and $\mathcal{R}$ are linear mappings we can conclude that

$$\begin{cases} \mathcal{L}, \mathcal{R} \in B(\mathcal{A}, B(\mathcal{A})) \\ \|\mathcal{R}\|_{B(\mathcal{A},B(\mathcal{A}))}, \|\mathcal{L}\|_{B(\mathcal{A},B(\mathcal{A}))} \leq 1. \end{cases} \tag{10}$$

In (6) we mean

$$(\exists\, Y_0 \subseteq B(G))(\mathcal{N}^* = \{\hat{A} \upharpoonright \mathcal{N} \mid A \in Y_0\}),$$

where $\hat{(\cdot)} : B(G) \to (B(G)^*)^*$ is the canonical isometric embedding of $B(G)$ into its bidual.

In the work the following generalizations of (4) are proved for the case when $R : \mathbf{D} \subset G \to G$ is an unbounded scalar type spectral operator in a complex Banach space G, in particular when $R : \mathbf{D} \subset H \to H$ is an unbounded self-adjoint operator in a complex Hilbert space H. Under the assumptions that $S : U \to \mathbb{C}$ is an analytic map on an open neighbourhood U of the spectrum $\sigma(R)$ of R such that there is $L > 0$ such that $]-L, L[\cdot U \subseteq U$ and

$$\widetilde{S}_t \in \mathfrak{L}_E^{\infty}(\sigma(R)),\ \widetilde{\left(\frac{d\,S}{d\,\lambda}\right)_t} \in \mathfrak{L}_E^{\infty}(\sigma(R))$$

for all $t \in]-L, L[$, where $(S)_t(\lambda) \doteqdot S(t\lambda)$ and $(\frac{d\,S}{d\,\lambda})_t(\lambda) \doteqdot \frac{d\,S}{d\,\lambda}(t\lambda)$ for all $t \in]-L, L[$ and $\lambda \in U$, while for any map $F : U \to \mathbb{C}$ we set $\widetilde{F}$ the $\mathbf{0}-$extension of F to $\mathbb{C}$. The following statements are proved.

(1) If

$$\int^{*} \left\| \widetilde{\left(\frac{d\,S}{d\,\lambda}\right)_t} \right\|_{\infty}^{E} d\,t < \infty \tag{11}$$

and for all $\omega \in \mathcal{N}$ the map $]-L, L[\ni t \mapsto \omega\left(\frac{d\,S}{d\,\lambda}(tR)\right) \in \mathbb{C}$ is Lebesgue measurable, then in Corollary 2.33 it is proved that formula (4) holds where the integral is the weak-integral [6] with respect to the Lebesgue measure and with respect to the $\sigma(B(G), \mathcal{N})-$topology for any $E-$appropriate set $\mathcal{N}$ with the duality property. Moreover in Corollary 2.34 it is proved that formula (4) also holds when $\widetilde{\left(\frac{d\,S}{d\,\lambda}\right)_t} \in \mathfrak{L}_E^{\infty}(\sigma(R))$ almost everywhere on $]-L, L[$ with respect to the Lebesgue measure.

(2) In particular it is proved that formula (4) holds where the integral is the weak-integral with respect to the Lebesgue measure and with respect to the sigma-weak operator topology, when G is a Hilbert space (Corollary 2.35).

(3) If in addition to (11), G is a reflexive complex Banach space then in Corollary 2.36 it is proved that formula (4) holds where the integral is the weak-integral with respect to the Lebesgue measure and with respect to the weak operator topology.

[6] See formula (21) below.

(4) If

$$\sup_{t\in]-L,L[}\left\|\widetilde{\left(\frac{dS}{d\lambda}\right)}_t\right\|_\infty^E<\infty, \tag{12}$$

then in Theorem 1.25 it is proved that formula (4) holds where the integral is with respect to the Lebesgue measure and with respect to the strong operator topology.

(5) In Theorem 1.23 it is proved that if in addition to the (12)

$$\sup_{t\in]-L,L[}\|\widetilde{(S)}_t\|_\infty^E<\infty,$$

then for all $v\in\mathbf{D}$ the mapping $]-L,L[\ni t\mapsto S(tR)v\in G$ is differentiable, and $(\forall v\in\mathbf{D})(\forall t\in]-L,L[)$

$$\frac{dS(tR)v}{dt}=R\frac{dS}{d\lambda}(tR)v. \tag{13}$$

(6) In Corollary 1.27 formula (3) is deduced from formula (4) for any unbounded scalar type spectral operator $T:\mathbf{D}(T)\subset G\to G$ in a complex Banach space G with real spectrum.

In these statements $\frac{dS}{d\lambda}(tR)$ and $S(tR)$ are understood in the framework of the Borel functional calculus for unbounded scalar type spectral operators in G. See definition $18.2.10.$ in Vol 3 of the Dunford-Schwartz monograph [**DS**] (also see Section 1 of the work).

In order to prove equality (4) when R is an unbounded scalar type spectral operator in G, we procede in two steps. First of all we consider the Banach spaces $G_{\sigma_n}\doteqdot E(\sigma_n)G$ where $\sigma_n\doteqdot B_n(\mathbf{0})\subset\mathbb{C}$, with $n\in\mathbb{N}$, the bounded operators $R_{\sigma_n}\doteqdot RE(\sigma_n)$, and their restrictions $(R_{\sigma_n}\upharpoonright G_{\sigma_n})$ to G_{σ_n}. Then by Key Lemma 1.7 the operators $R_{\sigma_n}\upharpoonright G_{\sigma_n}$ are bounded *scalar type spectral* operators on G_{σ_n}, and for all $x\in G$

$$S(R)x=\lim_{n\in\mathbb{N}}S(R_{\sigma_n}\upharpoonright G_{\sigma_n})E(\sigma_n)x, \tag{14}$$

in G and

(15)
$$(R_{\sigma_n}\upharpoonright G_{\sigma_n})\int_{u_1}^{u_2}\frac{dS}{d\lambda}(t(R_{\sigma_n}\upharpoonright G_{\sigma_n}))\,dt=S(u_2(R_{\sigma_n}\upharpoonright G_{\sigma_n}))-S(u_1(R_{\sigma_n}\upharpoonright G_{\sigma_n})).$$

The second and most important step it is to set up a *limiting* procedure, which allows by using the convergence (14) to extend the "local" equality (15) to the "global" one (4).

As we shall see below such a limiting procedure can be set up in a very general context. First we wish to point out that the following equalities for all $n\in\mathbb{N}$ and $t\in]-L,L[$, which follow from Key Lemma 1.7 are essential for making this limiting procedure possible

$$\begin{cases}\frac{dS}{d\lambda}(tR)E(\sigma_n)=\frac{dS}{d\lambda}(t(R_{\sigma_n}\upharpoonright G_{\sigma_n}))E(\sigma_n),\\ S(tR)E(\sigma_n)=S(t(R_{\sigma_n}\upharpoonright G_{\sigma_n}))E(\sigma_n).\end{cases} \tag{16}$$

We note that one cannot replace in (15) $R_{\sigma_n}\upharpoonright G_{\sigma_n}$ with the simpler operator R_{σ_n} for the following reason. Although R_{σ_n} is a bounded operator on G for $n\in\mathbb{N}$ and $Rx=\lim_{n\in\mathbb{N}}R_{\sigma_n}x$ in G, in general R_{σ_n} is not a scalar type spectral operator, hence the expression $\frac{dS}{d\lambda}(tR_{\sigma_n})$ is not defined in the Dunford-Schwartz Functional Calculus for scalar type spectral operators, which turns to be mandatory in the sequel when using general Borelian maps not necessarily analytic.

Next we formulate a rather general statement allowing, by using a limiting procedure, to pass from "local" equalities similar to (15) to "global" ones similar to (4).

We generalize (4) in several directions. We replace

- the operator R to the left of the integral by a function $g(R)$, where g is a general Borelian map on $\sigma(R)$ [7],
- the compact set $[u_1, u_2]$ and the Lebesgue measure on it by a general locally compact space X and a complex Radon measure on it respectively,
- the map $[u_1, u_2] \ni t \to \left(\frac{dS}{d\lambda}\right)_t \in Bor(\sigma(R))$ by the map $X \ni x \to f_x \in Bor(\sigma(R))$ such that $\widetilde{f_x} \in \mathfrak{L}_E^\infty(\sigma(R))$ where $\widetilde{f_x}$ is the $\mathbf{0}-$extension to $\mathbb{C}$ of f_x, and the map $X \ni x \to f_x(R) \in B(G)$ is strongly integrable with respect to the measure μ; [8]
- the map $S_{u_2} - S_{u_1}$ by a Borelian map h on $\sigma(R)$ such that $\widetilde{h} \in \mathfrak{L}_E^\infty(\sigma(R))$.

One of the main results of the work is Theorem 1.18 where we prove that if $\{\sigma_n\}_{n\in\mathbb{N}}$ is an $E-$sequence [9], and [10] for all $n \in \mathbb{N}$

$$R_{\sigma_n} \upharpoonright G_{\sigma_n} \doteqdot RE(\sigma_n) \upharpoonright (G_{\sigma_n} \cap Dom(R)), \tag{17}$$

and for all $n \in \mathbb{N}$ the following *local* inclusion

$$g(R_{\sigma_n} \upharpoonright G_{\sigma_n}) \int f_x(R_{\sigma_n} \upharpoonright G_{\sigma_n})\, d\,\mu(x) \subseteq h(R_{\sigma_n} \upharpoonright G_{\sigma_n}) \tag{18}$$

holds, then $h(R) \in B(G)$ and the *global* equality holds, i.e.

$$g(R) \int f_x(R)\, d\,\mu(x) = h(R). \tag{19}$$

Here all the integrals are with respect to the strong operator topology.

Now we can describe Extension Theorem and the Newton-Leibnitz formula for the integration with respect to the $\sigma(B(G), \mathcal{N})-$ topology, where $\mathcal{N}$ is a suitable subset of $B(G)^*$, which, roughly speaking, is the weakest among reasonable locally convex topologies on $B(G)$, for which the aforementioned limiting procedure can be performed.

In Section 2 we recall the definition of scalar essential $\mu-$integrability and the weak-integral of maps defined on X and with values in a Hausdorff locally convex spaces, where μ is a Radon measure on a locally compact space X.

Here we need just to apply these definitions to the case of $\sigma(B(G), \mathcal{N})$, i.e. the weak topology on $B(G)$ defined by the standard duality between $B(G)$ and $\mathcal{N}$ where $\mathcal{N}$ is a subset of the (topological) dual $B(G)^*$ of $B(G)$ such that it separates the points of $B(G)$.

Thus $f : X \to \langle B(G), \sigma(B(G), \mathcal{N})\rangle$ is by definition scalarly essentially $\mu-$integrable or equivalently $f : X \to B(G)$ is scalarly essentially $\mu-$integrable with respect to the measure μ and with respect to the $\sigma(B(G), \mathcal{N})$ topology on $B(G)$ if for all $\omega \in \mathcal{N}$ the map $\omega \circ f : X \to \mathbb{C}$ is essentially $\mu-$integrable [11], so we can define its *integral* as the

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[7] The most interesting case is when the operator $g(R)$ is unbounded.

[8] This means that $X \ni x \to f_x(R)v \in G$ is integrable with respect to the measure μ for all $v \in G$, in the sense of Ch 4, §4 of Bourbaki [**INT**], and the map $G \ni v \mapsto \int f_x(R)v \in G$ is a (linear) bounded operator on G.

[9] By definition this means that for all $n \in \mathbb{N}$ $\sigma_n \in \mathcal{B}(\mathbb{C})$, for all $n, m \in \mathbb{N}$ $n > m \Rightarrow \sigma_n \supseteq \sigma_m$; supp$(E) \subseteq \bigcup_{n\in\mathbb{N}} \sigma_n$; hence we have $\lim_{n\in\mathbb{N}} E(\sigma_n) = \mathbf{1}$ strongly.

[10] By Key Lemma 1.7 $R_{\sigma_n} \upharpoonright G_{\sigma_n}$ is a scalar type spectral operator in the complex Banach space G_{σ_n}, but in contrast to the previous case where $\sigma_n \doteqdot B_n(\mathbf{0})$ was bounded, here σ_n could be unbounded so it may happen that $G_{\sigma_n} \nsubseteq Dom(R)$ hence the restriction $R_{\sigma_n} \upharpoonright G_{\sigma_n}$ of R_{σ_n} to G_{σ_n} has to be defined on the set $G_{\sigma_n} \cap Dom(R)$, and it could be an unbounded operator in G_{σ_n}

[11] See for the definition Ch. 5, §1, $n^\circ 3$, of [**INT**]

following linear operator

$$\mathcal{N} \ni \omega \mapsto \int \omega(f(x)) d\,\mu(x) \in \mathbb{C}.$$

Let $f : X \to \langle B(G), \sigma(B(G), \mathcal{N})\rangle$ be scalarly essentially $\mu-$integrable and assume that

$$(\exists\, B \in B(G))(\forall \omega \in \mathcal{N}) \left(\omega(B) = \int \omega(f(x)) d\,\mu(x)\right). \tag{20}$$

Notice that the operator B is defined by this condition uniquely. In this case, by definition $f : X \to \langle B(G), \sigma(B(G), \mathcal{N})\rangle$ *is scalarly essentially* $(\mu, B(G))-$*integrable* or $f : X \to B(G)$ *is scalarly essentially* $(\mu, B(G))-$*integrable with respect to the* $\sigma(B(G), \mathcal{N})-$ *topology* and its *weak-integral with respect to the measure* μ *and with respect to the* $\sigma(B(G), \mathcal{N})-$ *topology* or simply its **weak-integral**, is defined by

$$\int f(x) d\,\mu(x) \doteqdot B. \tag{21}$$

Next we can state **Theorem 2.25** , the main result of the work.

THEOREM 0.1 ($\sigma(B(G), \mathcal{N})-$ **Extension Theorem**). *Let G be a complex Banach space, X a locally compact space and μ a complex Radon measure on it. In addition let R be a possibly unbounded scalar type spectral operator in G, $\sigma(R)$ its spectrum, E its resolution of the identity and $\mathcal{N}$ an $E-$appropriate set. Let the map $X \ni x \mapsto f_x \in Bor(\sigma(R))$ be such that $\widetilde{f_x} \in \mathfrak{L}_E^\infty(\sigma(R))$ $\mu-$ locally almost everywhere on X and the map $X \ni x \mapsto f_x(R) \in \langle B(G), \sigma(B(G), \mathcal{N})\rangle$ be scalarly essentially $(\mu, B(G))-$integrable. Finally let $g, h \in Bor(\sigma(R))$ and $\widetilde{h} \in \mathfrak{L}_E^\infty(\sigma(R))$.*

If $\{\sigma_n\}_{n\in\mathbb{N}}$ is an $E-$sequence and for all $n \in \mathbb{N}$

$$g(R_{\sigma_n} \upharpoonright G_{\sigma_n}) \int f_x(R_{\sigma_n} \upharpoonright G_{\sigma_n})\, d\,\mu(x) \subseteq h(R_{\sigma_n} \upharpoonright G_{\sigma_n}) \tag{22}$$

then $h(R) \in B(G)$ and

$$g(R) \int f_x(R)\, d\,\mu(x) = h(R). \tag{23}$$

In (22) *the weak-integral is with respect to the measure μ and with respect to the $\sigma(B(G_{\sigma_n}), \mathcal{N}_{\sigma_n})-$ topology* [12]*, while in* (23) *the weak-integral is with respect to the measure μ and with respect to the $\sigma(B(G), \mathcal{N})-$ topology.*

Notice that $g(R)$ is a possibly **unbounded** operator in G.

We list the most important results that allow to prove Theorem 2.25:

(1) Key Lemma 1.7;
(2) "Commutation" property (Theorem 2.13):

$$\forall \sigma \in \mathcal{B}(\mathbb{C}) \left[\int f_x(R) d\,\mu(x),\; E(\sigma)\right] = \mathbf{0}; \tag{24}$$

(3) "Restriction" property (Theorem 2.22): for all $\sigma \in \mathcal{B}(\mathbb{C})$ we have that $f_x(R_\sigma \upharpoonright G_\sigma) \in B(G_\sigma)$, $\mu-$ locally almost everywhere on X, $X \ni x \mapsto f_x(R_\sigma \upharpoonright G_\sigma) \in \langle B(G_\sigma), \sigma(B(G_\sigma), \mathcal{N}_\sigma)\rangle$ is scalarly essentially $(\mu, B(G_\sigma))-$integrable, and

$$\int f_x(R_\sigma \upharpoonright G_\sigma)\, d\mu(x) = \int f_x(R)\, d\mu(x) \upharpoonright G_\sigma; \tag{25}$$

[12] $\mathcal{N}_{\sigma_n}$ is, roughly speaking, the set of the restrictions to $B(G_{\sigma_n})$ of all the functionals belonging to $\mathcal{N}$. For the exact definition and properties see Definition 2.20 and Lemma 2.17.

(4) finally the fact that

$$Dom\left(g(R)\int f_x(R)\,d\,\mu(x)\right) \text{ is dense in } G.$$

We remark that the reason for introducing the concept of an $E-$appropriate set is primarily for obtaining the commutation and restriction properties.

Now we define

$$\mathcal{N}_{st}(G) \doteqdot \langle B(G), \tau_{st}(G)\rangle^* = \mathfrak{L}_{\mathbb{C}}(\{\psi_{(\phi,v)} \mid (\phi, v) \in G^* \times G\}). \tag{26}$$

Here $\langle B(G), \tau_{st}(G)\rangle^*$ is the topological dual of $B(G)$ with respect to the strong operator topology, $\psi_{(\phi,v)} : B(G) \ni T \mapsto \phi(Tv) \in \mathbb{C}$, while $\mathfrak{L}_{\mathbb{C}}(J)$ is the complex linear space generated by the set $J \subseteq B(G)^*$. Then $\sigma(B(G), \mathcal{N}_{st}(G))$ is the weak operator topology on $B(G)$ and $\mathcal{N}_{st}(G)$ is an $E-$appropriate set for any spectral measure E.

Moreover we set in the case in which G is a complex Hilbert space

$$\mathcal{N}_{pd}(G) \doteqdot \text{ predual of } B(G),$$

which is by definition the linear space of all sigma-weakly continuous linear functionals on $B(G)$.

Note that

$$\mathcal{N}_{pd}(G)^* = B(G). \tag{27}$$

(See statement (iii) of Theorem $2.6.$, Ch. 2 of [**Tak**], or Proposition $2.4.18$ of [**BR**]). Here we mean that the normed subspace $\mathcal{N}_{pd}(G)^*$ of the bidual $(B(G)^*)^*$ is isometric to $B(G)$, through the canonical embedding of $B(G)$ into $(B(G)^*)^*$.

Hence we can apply the Extension Theorem 2.25 to the case $\mathcal{N} \doteqdot \mathcal{N}_{st}(G)$, or $\mathcal{N} \doteqdot \mathcal{N}_{pd}(G)$ and use the following additional property which is proved in Proposition 2.23

$$(\mathcal{N}_{st}(G))_\sigma = \mathcal{N}_{st}(G_\sigma), \text{ and } (\mathcal{N}_{pd}(G))_\sigma = \mathcal{N}_{pd}(G_\sigma). \tag{28}$$

The reason of introducing the concept of duality property for $E-$appropriate set is primarily for assuring that a map $f : X \to \langle B(G), \sigma(B(G), \mathcal{N})\rangle$ scalarly essentially $\mu-$integrable is also $(\mu, B(G))-$integrable.

As an application of this fact and of the Extension theorem we obtain the Newton-Leibnitz formula in (4) by replacing $\mathcal{A}$ with $B(G)$, R with an unbounded scalar type spectral operator in a complex Banach space G, by considering S analytic in an open neighbourhood U of $\sigma(R)$ such that $]-L, L[\cdot U \subseteq U$, and the integral with respect to the $\sigma(B(G), \mathcal{N})-$topology, where $\mathcal{N}$ is an $E-$appropriate set with the duality property (**Corollary 2.33**).

Finally in a similar way we obtain the corresponding results for the cases of the sigma-weak operator topology (Corollary 2.35), and for the cases of weak operator topology (Corollary 2.36). The last result is a complement to Theorem 1.25.

Summary of the main results

Let G be a complex Banach space, R an unbounded scalar type spectral operator in G, for example an unbounded self-adjoint operator in a Hilbert space, $\sigma(R)$ its spectrum and E its resolution of identity. The **main results** of the work are the following ones.

(1) Extension procedure leading from local equality (22) to global equality (23) for integration with respect to the $\sigma(B(G), \mathcal{N})-$topology (Theorem 2.25 if $\mathcal{N}$ is an $E-$appropriate set and Corollary 2.26 if $\mathcal{N}$ is an $E-$appropriate set with the duality property).

(2) Extension procedure leading from local equality (22) to global equality (23) for integration with respect to the sigma-weak topology (Corollary 2.28 and Theorem 2.29) and for integration with respect to the weak operator topology (Corollary 2.27 and Theorem 2.30 or Theorem 1.18 and Corollary 1.19).

(3) Newton-Leibnitz formula (4) for a suitable analytic map S for integration with respect to the $\sigma(B(G), \mathcal{N})-$ topology, where $\mathcal{N}$ is an $E-$appropriate set with the duality property (Corollary 2.33 and Corollary 2.34); for integration with respect to the sigma-weak topology (Corollary 2.35) and for integration with respect to the weak operator topology (Corollary 2.36 and Theorem 1.25).

(4) Differentiation formula (13) for a suitable analytic map S (Theorem 1.21 and Theorem 1.23).

(5) A new proof for the resolvent formula (3) via formula (4) (Corollary 1.27).

CHAPTER 1

Extension theorem. The case of the strong operator topology

1. Key lemma

PRELIMINARIES 1.1. **Integrals of bounded Borelian functions with respect to a vector valued measure.** In the sequel $G \doteqdot \langle G, \|\cdot\|_G \rangle$ will be a complex Banach space. Denote by $\mathrm{Pr}(G)$ the class of all projectors on G, that is the class of $P \in B(G)$ such that $P^2 = P$. Consider a Boolean algebra $\mathcal{B}_X$, see Sec. 1.12 of [**DS**], of subsets of a set X, with respect to the order relation defined by $\sigma \geq \delta \Leftrightarrow \sigma \supseteq \delta$ and complemented by the operation $\sigma' \doteqdot \complement\sigma$. In particular $\mathcal{B}_X$ contains $\emptyset$ and X and is closed under finite intersection and finite union.

The map $E : \mathcal{B}_X \to B(G)$ is called a spectral measure in G on $\mathcal{B}_X$, or simply on X if X is a topological space and $\mathcal{B}_X$ is the Boolean algebra of its Borelian subsets, if

(1) $E(\mathcal{B}_X) \subseteq \mathrm{Pr}(G)$;
(2) $(\forall \sigma_1, \sigma_2 \in \mathcal{B}_X)(E(\sigma_1 \cap \sigma_2) = E(\sigma_1)E(\sigma_2))$;
(3) $(\forall \sigma_1, \sigma_2 \in \mathcal{B}_X)(E(\sigma_1 \cup \sigma_2) = E(\sigma_1) + E(\sigma_2) - E(\sigma_1)E(\sigma_2))$;
(4) $E(X) = \mathbf{1}$;
(5) $E(\emptyset) = \mathbf{0}$.

(See Definition 15.2.1 of [**DS**]).

If condition (3) is replaced by condition

$$(3')(\forall \sigma_1, \sigma_2 \in \mathcal{B}_X \mid \sigma_1 \cap \sigma_2 = \emptyset)(E(\sigma_1 \cup \sigma_2) = E(\sigma_1) + E(\sigma_2)),$$

we obtain an equivalent definition.

Notice that if E is a spectral measure in G on $\mathcal{B}_X$, then it is a Boolean homomorphism onto the Boolean algebra $E(\mathcal{B}_X)$ with respect to the order relation induced by that defined in $\mathrm{Pr}(G)$ by $P \geq Z \Leftrightarrow Z = ZP$ and complemented by the operation $P' \doteqdot (\mathbf{1} - P)$. Indeed for all $\sigma, \delta \in \mathcal{B}_X$ we have $\delta \subseteq \sigma \Rightarrow E(\delta) = E(\delta \cap \sigma) \doteq E(\delta)E(\sigma) \Leftrightarrow E(\delta) \leq E(\sigma)$, while $\mathbf{1} = E(\sigma \cup \complement\sigma) = E(\sigma) + E(\complement\sigma)$.

A spectral measure E is called *(weakly) countable additive* if for all sequences $\{\varepsilon_n\}_{n\in\mathbb{N}} \subset \mathcal{B}_X$ of disjoint sets, for all $x \in G$ and for all $\phi \in G^*$ we have

$$\phi\left(E(\bigcup_{n\in\mathbb{N}} \varepsilon_n)x\right) = \sum_{n=1}^{\infty} \phi\left(E(\varepsilon_n)x\right).$$

If $\mathcal{B}_X$ is a σ-field, i.e. a Boolean algebra closed under the operation of forming countable unions, we have by Corollary 15.2.4. of the [**DS**] that E is countably additive with respect to the strong operator topology, i.e. for all sequence $\{\varepsilon_n\}_{n\in\mathbb{N}} \subset \mathcal{B}(\mathbb{C})$ of disjoint sets and

for all $x \in G$ we have [1]

$$E(\bigcup_{n\in\mathbb{N}} \varepsilon_n)x = \sum_{n=1}^{\infty} E(\varepsilon_n)x = \sum_{n\in\mathbb{N}} E(\varepsilon_n)x. \tag{29}$$

Since $E(\bigcup_{n\in\mathbb{N}} \varepsilon_n) = E(\bigcup_{n\in\mathbb{N}} \varepsilon_{\rho(n)})$, for any permutation ρ of $\mathbb{N}$, hence $\sum_{n=1}^{\infty} E(\varepsilon_n)x = \sum_{n=1}^{\infty} E(\varepsilon_{\rho(n)})x$ for all $x \in G$, therefore by Proposition 9,§$5.7.$, Ch. 3 of [**GT**] we obtain the second equality in (29). By $\mathcal{B}(\mathbb{C})$ we denote the set of the Borelian subsets of $\mathbb{C}$, and by $Bor(U)$ the complex linear space of all Borelian complex maps defined on a Borelian subset U of $\mathbb{C}$. We denote with $\mathbf{TM}$ the space of the totally $\mathcal{B}(\mathbb{C})-$ measurable maps [2], which is the closure in the Banach space $\langle \mathbf{B}(\mathbb{C}), \|\cdot\|_{\sup}\rangle$ of all complex bounded functions on $\mathbb{C}$ with respect to the norm $\|g\|_{\sup} \doteqdot \sup_{\lambda\in\mathbb{C}} |g(\lambda)|$, of the linear space generated by the set $\{\chi_\sigma \mid \sigma \in \mathcal{B}(\mathbb{C})\}$, where χ_σ is the characteristic function of the set σ. $\langle \mathbf{TM}, \|\cdot\|_{\sup}\rangle$ is a Banach space, and the space of all bounded Borelian complex functions is in $\mathbf{TM}$ so dense in it. Finally $\langle \mathbf{TM}, \|\cdot\|_{\sup}\rangle$ is a C^*-subalgebra, in particular a Banach subalgebra, of $\langle \mathbf{B}(\mathbb{C}), \|\cdot\|_{\sup}\rangle$ if we define the pointwise operations of product and involution on $\mathbf{B}(\mathbb{C})$.

Let X be a complex Banach space and $F : \mathcal{B}(\mathbb{C}) \to X$ a weakly countably finite additive vector valued measure, see Section $4.10.$ of [**DS**], then we can define the integral with respect to F, see Section 10.1 of [**DS**], which will be denoted by $\int_{\mathbb{C}} f\,dF$. The operator

$$\mathbf{I}_{\mathbb{C}}^F : \mathbf{TM} \ni f \mapsto \int_{\mathbb{C}} f\,dF \in X \tag{30}$$

is linear and norm-continuous [3]. We have the following useful property if Y is a $\mathbb{C}-$Banach space and $Q \in B(X, Y)$, then

$$Q \circ \mathbf{I}_{\mathbb{C}}^F = \mathbf{I}_{\mathbb{C}}^{Q\circ F}, \tag{31}$$

see statement (f) of Theorem $4.10.8.$ of the [**DS**].

If $X \doteqdot B(G)$, the case we are mostly interested in, we have, as an immediate result of this property and the fact that the map $Q_x : B(G) \ni A \mapsto Ax \in G$ is linear and continuous for all $x \in G$, that

$$(\forall x \in G)(\forall f \in \mathbf{TM})(\mathbf{I}_{\mathbb{C}}^F(f)x = \mathbf{I}_{\mathbb{C}}^{F^x}(f)). \tag{32}$$

Here $F^x : \mathcal{B}(\mathbb{C}) \ni \sigma \mapsto F(\sigma)x$. Finally if E is a spectral measure on $\mathbb{C}$, then $\mathbf{I}_{\mathbb{C}}^E$ is a continuous unital homomorphism between the two Banach algebras $\langle \mathbf{TM}, \|\cdot\|_{\sup}\rangle$, and $\left\langle B(G), \|\cdot\|_{B(G)}\right\rangle$ and $\mathbf{I}_{\mathbb{C}}^E(\chi_{supp\ E}) = \mathbf{1}$, see (34) and Section (2), Ch 15 of [**DS**].

Borel functional calculus for possibly unbounded scalar type spectral operators in G. If $T : \mathcal{D}(T) \subseteq G \to G$ is a possibly unbounded linear operator then we denote by $\sigma(T)$ its standard spectrum. A possibly unbounded linear operator $T : \mathcal{D}(T) \subseteq G \to G$ is called a **spectral operator in** G if it is closed and there exists a countably additive spectral measure $E : \mathcal{B}(\mathbb{C}) \to \Pr(G)$ such that

[1] By definition, see Ch.3 of [**GT**], $v = \sum_{n\in\mathbb{N}} E(\varepsilon_n)x$ if $v = \lim_{J\in\mathcal{P}_\omega(\mathbb{N})} \sum_{n\in J} E(\varepsilon_n)x$, where $\mathcal{P}_\omega(\mathbb{N})$ is the direct ordered set of all finite subsets of $\mathbb{N}$ ordered by inclusion.

[2] In [**DS**] denoted by $B(\mathbb{C}, \mathcal{B}(\mathbb{C}))$, while by using the notations of [**Din2**] and considering $\mathbb{C}$ as a real Banach space we have $\mathbf{TM} = \mathbf{TM}_{\mathbb{R}}(\mathcal{B}(\mathbb{C}))$.

[3] Notice that if we identify $B(G)$ with $B(\mathbb{R}, B(G))$ and recall that $\mathbf{TM} = \mathbf{TM}_{\mathbb{R}}(\mathcal{B}(\mathbb{C}))$, then with the notations of Definition 24, §1, Ch. 1 of [**Din2**] we have that $\mathbf{I}_{\mathbb{C}}^E$ is the immediate integral with respect to the vector valued measure $E : \mathcal{B}(\mathbb{C}) \to B(\mathbb{R}, B(G))$.

i: for all bounded sets $\delta \in \mathcal{B}(\mathbb{C})$

$$E(\delta)G \subseteq \mathcal{D}(T);$$

ii: $(\forall \delta \in \mathcal{B}(\mathbb{C}))(\forall x \in \mathcal{D}(T))$ we have

(1) $(E(\delta)\mathcal{D}(T) \subseteq \mathcal{D}(T))$,

(2) $TE(\delta)x = E(\delta)Tx$;

iii: for all $\delta \in \mathcal{B}(\mathbb{C})$ we have

$$\sigma\left(T \upharpoonright (\mathcal{D}(T) \cap E(\delta)G)\right) \subseteq \overline{\delta}.$$

Here $\sigma\left(T \upharpoonright (\mathcal{D}(T) \cap E(\delta)G)\right)$ is the spectrum of the restriction of T to the domain $\mathcal{D}(T) \cap E(\delta)G$.

(See Definition $18.2.1.$ of the [**DS**]). We call any E with the above properties a **resolution of the identity of** T. Theorem $18.2.5.$ of [**DS**] states that the resolution of the identity of a spectral operator is unique.

Finally we call support of a spectral measure E on $\mathcal{B}_X$, the following set

$$\operatorname{supp} E \doteqdot \bigcap_{\{\sigma \in \mathcal{B}_X \,|\, E(\sigma) = \mathbf{1}\}} \overline{\sigma}.$$

It is easy to see [4] that

$$E(\operatorname{supp} E) = \mathbf{1}. \tag{34}$$

Notice that an unbounded spectral operator T is closed by definition. Now we will show that T is also densely defined. In fact if E is the resolution of the identity of T and if $\{\sigma_n\}_{n \in \mathbb{N}} \subset \mathcal{B}(\mathbb{C})$ is a non decreasing sequence of Borelian sets such that $\sigma(T) \subseteq \bigcup_{n \in \mathbb{N}} \sigma_n$, then by the strong countable additivity of E, the fact that $E(\sigma(T)) = \mathbf{1}$ we can deduce $\mathbf{1} = \lim_{n \in \mathbb{N}} E(\sigma_n)$ in the strong operator topology of $B(G)$, see (47). Now we can choose $\{\sigma_n\}_{n \in \mathbb{N}}$ such that $\sigma_n \doteqdot B_n(\mathbf{0}) \doteqdot \{\lambda \in \mathbb{C} \mid |\lambda| < n\}$, or $\sigma_n \doteqdot W(\mathbf{0}, 2n) \doteqdot \{\lambda \in \mathbb{C} \mid |Re(\lambda)| < n, |Im(\lambda)| < n,\}$. But by the property (i) of the Definition $18.2.1.$ of [**DS**], we know that for all bounded sets $\sigma \in \mathcal{B}(\mathbb{C})$ we have $E(\sigma)G \subseteq Dom(T)$. Therefore we conclude that for all $v \in G$, $v = \lim_{n \in \mathbb{N}} E(\sigma_n)v$ and for all $n \in \mathbb{N}$, $E(\sigma_n)v \in Dom(T)$, so $Dom(T)$ is dense in G.

We want to remark that for each possibly unbounded spectral operator T in G by denoting with $\sigma(T)$ its spectrum and with $E : \mathcal{B}(\mathbb{C}) \to \Pr(G)$ its resolution of the identity, we deduce by Lemma $18.2.25.$ of [**DS**] that $\sigma(T)$ is closed, that $\operatorname{supp} E = \sigma(T)$ so by (34)

$$E(\sigma(T)) = \mathbf{1}.$$

Now we will give the definition of the Borel functional calculus for unbounded spectral operators in a complex Banach space G, that is essentially the same as in Definition $18.2.10.$ of the [**DS**].

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[4] Indeed let $S \doteqdot \operatorname{supp} E$ then

$$\complement S = \bigcup_{\{\sigma \in \mathcal{B}_X \,|\, E(\sigma) = \mathbf{1}\}} \complement \overline{\sigma}. \tag{33}$$

Moreover E is order-preserving so for all $\sigma \in \mathcal{B}_X$ such that $E(\sigma) = \mathbf{1}$ we have $E(\complement \overline{\sigma}) \leq E(\complement \sigma) = \mathbf{1} - E(\sigma) = \mathbf{0}$. Hence by the definition of the order $E(\complement \overline{\sigma}) = E(\complement \overline{\sigma})\mathbf{0} = \mathbf{0}$. Therefore by the Principle of localization (Corollary, Ch 3, §2, $n^\circ 1$ of [**INT**]) which holds also for vector measures (footnote in Ch 6, §2, $n^\circ 1$ of [**INT**]) we deduce by (33) that $E(\complement S) = \mathbf{0}$. Finally

$$E(S) = \mathbf{1} - E(\complement S) = \mathbf{1}.$$

DEFINITION 1.2. Let X be a set, $S \subset X$, V a vector space over $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ and $f : S \to V$. Then we define $\widetilde{f}^X$, or simply $\widetilde{f}$ when it doesn't cause confusion, to be the $\mathbf{0}-$extension of f to X, i.e. $\widetilde{f} : X \to V$ such that $\widetilde{f} \upharpoonright S = f$ and $\widetilde{f}(x) = \mathbf{0}$ for all $x \in (X - S)$, where $\mathbf{0}$ is the zero vector of V.

DEFINITION 1.3. [Borel Functional Calculus of E] Assume that

(1) $E : \mathcal{B}(\mathbb{C}) \to \Pr(G)$ is a countably additive spectral measure and S its support;
(2) $f \in Bor(S)$;
(3) for all $\sigma \subseteq \mathbb{C}$ we set $f_\sigma : \mathbb{C} \to \mathbb{C}$ such that $f_\sigma \doteqdot \widetilde{f} \cdot \chi_\sigma$;
(4) $\delta_n \doteqdot [-n, +n]$ and
$$f_n \doteqdot f_{\overset{-1}{|f|}(\delta_n)}.$$
Here $(\forall \sigma \subseteq \mathbb{C})(\forall g : \mathcal{D} \to \mathbb{C})(\overset{-1}{g}(\sigma) \doteqdot \{\lambda \in \mathcal{D} \mid g(\lambda) \in \sigma\})$.

Of course $f_n \in \mathbf{TM}$ for all $n \in \mathbb{N}$ so we can define the following operator in G

$$\begin{cases} Dom(f(E)) \doteqdot \{x \in G \mid \exists \lim_{n\in\mathbb{N}} \mathbf{I}_{\mathbb{C}}^{E}(f_n)x\} \\ (\forall x \in Dom(f(E)))(f(E)x \doteqdot \lim_{n\in\mathbb{N}} \mathbf{I}_{\mathbb{C}}^{E}(f_n)x). \end{cases} \tag{35}$$

Here all limits are considered in the space G. We call the map $f \mapsto f(E)$ the **Borel functional calculus of the spectral measure** E.

In the case in which E is the resolution of the identity of a possibly unbounded spectral operator T, recalling Lemma 18.2.25. of [**DS**] stating that $\sigma(T)$ is the support of E, we can define $f(T) \doteqdot f(E)$ for any map $f \in Bor(\sigma(T))$ and call the map

$$Bor(\sigma(T)) \ni f \mapsto f(T)$$

the **Borel functional calculus of the operator** T.

DEFINITION 1.4. [18.2.12. of [**DS**]] A *spectral operator of scalar type in* G or a **scalar type spectral operator in** G is a possibly unbounded linear operator R in G such that there exists a countably additive spectral measure $E : \mathcal{B}(\mathbb{C}) \to \Pr(G)$ with support S and the property

$$R = \imath(E).$$

Here $\imath : S \ni \lambda \mapsto \lambda \in \mathbb{C}$, and $\imath(E)$ is relative to the Borel functional calculus of the spectral measure E. We call E a **resolution of the identity of** R.

Let R be a scalar type spectral operator in G and E a resolution of the identity of R, then we have the following statements by [**DS**]:

- T is a spectral operator in G;
- E is the resolution of the identity of T as spectral operator;
- E is unique.

DEFINITION 1.5 ([**DS**]). Let $E : \mathcal{B}(\mathbb{C}) \to Pr(G)$ be a countably additive spectral measure and $U \in \mathcal{B}(\mathbb{C})$, then the space of all $E-$*essentially bounded* maps is the following linear space

$$\mathfrak{L}_E^\infty(U) \doteqdot \left\{f : \mathbb{C} \to \mathbb{C} \mid \|f\chi_U\|_\infty^E < \infty\right\}.$$

Here $\chi_U : \mathbb{C} \to \mathbb{C}$ is the characteristic map of U which is by definition equal to 1 in U and 0 in $\complement U$, and for each map $F : \mathbb{C} \to \mathbb{C}$

$$\|F\|_\infty^E \doteqdot E - ess \sup_{\lambda\in\mathbb{C}} |F(\lambda)| \doteqdot \inf_{\{\delta\in\mathcal{B}(\mathbb{C}) \mid E(\delta)=\mathbf{1}\}} \sup_{\lambda\in\delta} |F(\lambda)|.$$

For a Borelian map $f : U \supset \sigma(R) \to \mathbb{C}$, with $U \in \mathcal{B}(\mathbb{C})$, we define $f(R)$ to be the operator $(f \upharpoonright \sigma(R))(R)$. Let $g : U \subseteq \mathbb{C} \to \mathbb{C}$ be a Borelian map. Then g is $E-$essentially bounded if

$$E - ess \sup_{\lambda \in U} |g(\lambda)| \doteqdot \|\widetilde{g}\|_{\infty}^{E} < \infty.$$

See Definition $17.2.6.$ of [**DS**]. One formula arising by statement (i) of the Spectral Theorem $18.2.11.$ of the [**DS**], which will be used many times in the work is the following: for all Borelian complex function $f : \sigma(R) \to \mathbb{C}$ and for all $\phi \in G^*$ and $y \in Dom(f(R))$

$$\phi\,(f(R)y) = \int_{\mathbb{C}} \widetilde{f}\, d\, E_{(\phi,y)}. \tag{36}$$

Here G^* is the topological dual of G, that is the normed space of all $\mathbb{C}-$linear and continuous functionals on G with the $\sup-$norm, and for all $\phi \in G^*$ and $y \in G$ we define $E_{(\phi,y)} : \mathcal{B}(\mathbb{C}) \ni \sigma \mapsto \phi(E(\sigma)y) \in \mathbb{C}$. Finally if $P \in \Pr(G)$ then $\left\langle P(G), \|\cdot\|_{P(G)} \right\rangle$, with $\|\cdot\|_{P(G)} \doteqdot \|\cdot\|_G \upharpoonright P(G)$, is a Banach space. In fact let $\{v_n\}_{n\in\mathbb{N}} \subset G$ be such that $v = \lim_{n\in\mathbb{N}} Pv_n$, in $\|\cdot\|_G$, so $P = P^2$ being continuous we have that $Pv = \lim_{n\in\mathbb{N}} P^2 v_n = \lim_{n\in\mathbb{N}} Pv_n \doteqdot v$, so $v \in P(G)$, then $P(G)$ is closed in $\langle G, \|\cdot\|_G\rangle$, hence $\left\langle P(G), \|\cdot\|_{P(G)} \right\rangle$ is a Banach space. If $E : \mathcal{B}_Y \to \Pr(G)$ is a spectral measure in G on $\mathcal{B}_Y$ and $\sigma \in \mathcal{B}_Y$, then we shall denote by G_σ^E or simply G_σ the complex Banach space $E(\sigma)G$, without expressing its dependence by E whenever it does not cause confusion. In addition for any Q possibly unbounded operator in G we define for all $\sigma \in \mathcal{B}_Y$ the following possibly unbounded operator operator in G

$$Q_\sigma \doteqdot QE(\sigma).$$

Finally we shall denote by $\mathcal{B}_b(\mathbb{C})$ the subclass of all bounded subsets of $\mathcal{B}(\mathbb{C})$.

DEFINITION 1.6. Let F be a $\mathbb{C}-$Banach space, $P \in \Pr(F)$ and $S : Dom(S) \subseteq F \to F$, then we define

$$SP \upharpoonright P(F) \doteqdot SP \upharpoonright (P(F) \cap Dom(SP)). \tag{37}$$

Notice that by the property $P^2 = P$ we have $P(F) \cap Dom(S) = P(F) \cap Dom(SP)$, and that

$$SP \upharpoonright P(F) = S \upharpoonright (P(F) \cap Dom(S)).$$

Moreover in the case in which $PS \subseteq SP$ then

$$SP \upharpoonright P(F) : P(F) \cap Dom(S) \to P(F).$$

That is $SP \upharpoonright P(F)$ is a linear operator in the Banach space $P(F)$. Let $E : \mathcal{B}_Y \to \Pr(G)$ be a spectral measure in G on $\mathcal{B}_Y$, $\sigma \in \mathcal{B}_Y$ and Q a possibly unbounded operator in G such that $E(\sigma)Q \subseteq QE(\sigma)$, then

$$Q_\sigma \upharpoonright G_\sigma : G_\sigma \cap Dom(Q) \to G_\sigma.$$

In particular if R is a possibly unbounded scalar type spectral operator in G, E its resolution of the identity and $f \in Bor(\sigma(R))$, then by statement (g) of Theorem $18.2.11$ of [**DS**], we have that for all $\sigma \in \mathcal{B}(\mathbb{C})$

$$E(\sigma)f(R) \subseteq f(R)E(\sigma).$$

Hence for all $\sigma \in \mathcal{B}(\mathbb{C})$

$$\begin{cases} R_\sigma \upharpoonright G_\sigma = R_\sigma \upharpoonright (G_\sigma \cap Dom(R)) = R \upharpoonright (G_\sigma \cap Dom(R)) \\ f(R)_\sigma \upharpoonright G_\sigma = f(R)_\sigma \upharpoonright (G_\sigma \cap Dom(f(R))) = f(R) \upharpoonright (G_\sigma \cap Dom(f(R))) \end{cases} \tag{38}$$

are linear operators in G_σ. Finally $E(\sigma(R)) = \mathbf{1}$ implies $E(\sigma) = E(\sigma \cap \sigma(R))$ for all $\sigma \in \mathcal{B}(\mathbb{C})$ so by (38)

$$\begin{cases} R_\sigma \upharpoonright G_\sigma = R_{\sigma\cap\sigma(R)} \upharpoonright G_{\sigma\cap\sigma(R)} \\ f(R)_\sigma \upharpoonright G_\sigma = f(R)_{\sigma\cap\sigma(R)} \upharpoonright G_{\sigma\cap\sigma(R)}. \end{cases} \tag{39}$$

LEMMA 1.7 (Key Lemma). *Let R be a possibly unbounded scalar type spectral operator in G, E its resolution of the identity, $\sigma(R)$ its spectrum and $f \in Bor(\sigma(R))$. Then for all $\sigma \in \mathcal{B}(\mathbb{C})$*

(1) *$R_\sigma \upharpoonright G_\sigma$ is a scalar type spectral operator in G_σ whose resolution of the identity $\widetilde{E}_\sigma$ is such that for all $\delta \in \mathcal{B}(\mathbb{C})$*
$$\widetilde{E}_\sigma(\delta) = E(\delta) \upharpoonright G_\sigma \in B(G_\sigma),$$
(2)
$$f(R)_\sigma \upharpoonright G_\sigma = f\left(R_\sigma \upharpoonright G_\sigma\right),$$
(3) *for all $g \in Bor(\sigma(R))$ such that $g(\sigma \cap \sigma(R))$ is bounded, we have that*
$$g(R)E(\sigma) = \mathbf{I}_{\mathbb{C}}^{E}(\widetilde{g} \cdot \chi_\sigma) \in B(G).$$

PROOF. Let $\sigma \in \mathcal{B}(\mathbb{C})$. By the fact that $E(\sigma \cap \delta) = E(\delta)E(\sigma) = E(\sigma)E(\delta)$ for all $\delta \in \mathcal{B}(\mathbb{C})$ and $E(\sigma) \upharpoonright G_\sigma = \mathbf{1}_\sigma$ the unity operator on G_σ, we have for all $\delta \in \mathcal{B}(\mathbb{C})$

$$\widetilde{E}_\sigma(\delta) = E(\sigma \cap \delta) \upharpoonright G_\sigma \in B(G_\sigma). \tag{40}$$

In particular $\widetilde{E}_\sigma : \mathcal{B}(\mathbb{C}) \to B(G_\sigma)$, moreover E is a countably additive spectral measure in G, so

(41) $\widetilde{E}_\sigma$ is a countably additive spectral measure in G_σ.

By Lemma 18.2.2. of [**DS**] $\widetilde{E}_\sigma$ is the resolution of identity of the spectral operator $R_\sigma \upharpoonright G_\sigma$ so by Lemma 18.2.25. of [**DS**] applied to $R_\sigma \upharpoonright G_\sigma$

(42) supp $\widetilde{E}_\sigma = \sigma(R_\sigma \upharpoonright G_\sigma)$.

Furthermore by (39) and (iii) of Definition 18.2.1. of [**DS**] we have $\sigma(R_\sigma \upharpoonright G_\sigma) \subseteq \overline{\sigma \cap \sigma(R)}$, then by the equality $\overline{\overline{\sigma} \cap \sigma(R)} = \overline{\sigma} \cap \sigma(R)$, we deduce

$$\sigma(R_\sigma \upharpoonright G_\sigma) \subseteq \overline{\sigma} \cap \sigma(R) \subseteq \sigma(R). \tag{43}$$

Hence (42) and (43) imply that the operator function $f(\widetilde{E}_\sigma)$ is well defined. For all $x \in Dom(f(R)_\sigma \upharpoonright G_\sigma)$

$$\begin{aligned}
(f(R)_\sigma \upharpoonright G_\sigma)x &= f(R)x && \text{by (38)}\\
&= \lim_{n\in\mathbb{N}} \mathbf{I}_{\mathbb{C}}^{E^x}(\widetilde{f} \cdot \chi_{\underset{|f|(\delta_n)}{-1}}) && \text{by (35), (32)}\\
&= \lim_{n\in\mathbb{N}} \mathbf{I}_{\mathbb{C}}^{\widetilde{E}_\sigma^x}(\widetilde{f} \cdot \chi_{\underset{|f|(\delta_n)}{-1}}) && \text{by } x \in G_\sigma\text{, (41)}\\
&= \lim_{n\in\mathbb{N}} \mathbf{I}_{\mathbb{C}}^{\widetilde{E}_\sigma}(\widetilde{f} \cdot \chi_{\underset{|f|(\delta_n)}{-1}})x && \text{by (32)}\\
&= f(\widetilde{E}_\sigma)x. && \text{by (35)}
\end{aligned}$$

So $f(R)_\sigma \upharpoonright G_\sigma \subseteq f(\widetilde{E}_\sigma)$. For all $x \in Dom(f(\widetilde{E}_\sigma))$

$$\begin{aligned} f(\widetilde{E}_\sigma)x &= \lim_{n\in\mathbb{N}} \mathbf{I}_{\mathbb{C}}^{\widetilde{E}_\sigma^x}(\widetilde{f}\cdot\chi_{\overset{-1}{|f|}(\delta_n)}) \text{ by (35), (32)}\\ &= \lim_{n\in\mathbb{N}} \mathbf{I}_{\mathbb{C}}^{E^x}(\widetilde{f}\cdot\chi_{\overset{-1}{|f|}(\delta_n)})\\ &= \lim_{n\in\mathbb{N}} \mathbf{I}_{\mathbb{C}}^{E}(\widetilde{f}\cdot\chi_{\overset{-1}{|f|}(\delta_n)})x \qquad \text{by (32)}\\ &= (f(R)_\sigma \upharpoonright G_\sigma)x. \qquad \text{by (35), (38)} \end{aligned}$$

So $f(\widetilde{E}_\sigma) \subseteq f(R)_\sigma \upharpoonright G_\sigma$, then

$$f(R)_\sigma \upharpoonright G_\sigma = f(\widetilde{E}_\sigma) \tag{44}$$

Therefore statement (1) follows by setting $f = \imath$, while statement (2) follows by statement (1) and (44). Let $g \in Bor(\sigma(R))$ such that $g(\sigma \cap \sigma(R))$ is bounded, then

$$(\exists\, n \in \mathbb{N})(\forall m > n)(\sigma \cap \overset{-1}{|g|}(\delta_m) = \sigma \cap \sigma(R)).$$

Next $E(\sigma(R)) = \mathbf{1}$, so $E(\sigma) = E(\sigma)E(\sigma(R)) = E(\sigma(R) \cap \sigma)$. Since $\mathbf{I}_{\mathbb{C}}^{E}$ is an algebra homomorphism, for all $m \in \mathbb{N}$

$$\begin{aligned} \mathbf{I}_{\mathbb{C}}^{E}(\widetilde{g}\cdot\chi_{\overset{-1}{|g|}(\delta_m)})E(\sigma) &= \mathbf{I}_{\mathbb{C}}^{E}(\widetilde{g}\cdot\chi_{\overset{-1}{|g|}(\delta_m)})E(\sigma\cap\sigma(R))\\ &= \mathbf{I}_{\mathbb{C}}^{E}(\widetilde{g}\cdot\chi_{\overset{-1}{|g|}(\delta_m)})\mathbf{I}_{\mathbb{C}}^{E}(\chi_{\sigma\cap\sigma(R)})\\ &= \mathbf{I}_{\mathbb{C}}^{E}(\widetilde{g}\cdot\chi_{\overset{-1}{|g|}(\delta_m)}\cdot\chi_{\sigma\cap\sigma(R)})\\ &= \mathbf{I}_{\mathbb{C}}^{E}(\widetilde{g}\cdot\chi_{\overset{-1}{|g|}(\delta_m)\cap\sigma\cap\sigma(R)})\\ &= \mathbf{I}_{\mathbb{C}}^{E}(\widetilde{g}\cdot\chi_{\overset{-1}{|g|}(\delta_m)\cap\sigma}). \end{aligned}$$

This equality implies that

$$(\exists\, n \in \mathbb{N})(\forall m > n)(\mathbf{I}_{\mathbb{C}}^{E}(\widetilde{g}\cdot\chi_{\overset{-1}{|g|}(\delta_m)})E(\sigma) = \mathbf{I}_{\mathbb{C}}^{E}(\widetilde{g}\cdot\chi_{\sigma\cap\sigma(R)})). \tag{45}$$

Furthermore

$$\begin{aligned} \mathbf{I}_{\mathbb{C}}^{E}(\widetilde{g}\cdot\chi_{\sigma\cap\sigma(R)}) &= \mathbf{I}_{\mathbb{C}}^{E}(\widetilde{g}\chi_\sigma\chi_{\sigma(R)})\\ &= \mathbf{I}_{\mathbb{C}}^{E}(\widetilde{g}\chi_\sigma)\mathbf{I}_{\mathbb{C}}^{E}(\chi_{\sigma(R)})\\ &= \mathbf{I}_{\mathbb{C}}^{E}(\widetilde{g}\chi_\sigma)E(\sigma(R))\\ &= \mathbf{I}_{\mathbb{C}}^{E}(\widetilde{g}\chi_\sigma). \end{aligned}$$

Therefore by (45)

$$(\exists\, n \in \mathbb{N})(\forall m > n)(\mathbf{I}_{\mathbb{C}}^{E}(\widetilde{g}\cdot\chi_{\overset{-1}{|g|}(\delta_m)})E(\sigma) = \mathbf{I}_{\mathbb{C}}^{E}(\widetilde{g}\cdot\chi_\sigma)). \tag{46}$$

Moreover by definition in (35) we have for all $x \in Dom(g(R))$ that

$$g(R)x \doteqdot \lim_{n\to\infty} \mathbf{I}_{\mathbb{C}}^{E}(\widetilde{g}\cdot\chi_{\overset{-1}{|g|}(\delta_n)})x$$

and $Dom(g(R))$ is the set of $x \in G$ such that such a limit exists; thus by (46) we can conclude that $E(\sigma)G \subseteq Dom(g(R))$ and $g(R)E(\sigma) = \mathbf{I}_{\mathbb{C}}^{E}(\widetilde{g}\cdot\chi_\sigma) \in B(G)$, which is statement (3). $\square$

COROLLARY 1.8. *Let R be a possibly unbounded scalar type spectral operator in G, and $f \in Bor(\sigma(R))$. Then for all $\sigma \in \mathcal{B}(\mathbb{C})$*

$$f(R)E(\sigma) = f\left(R_\sigma \upharpoonright G_\sigma\right)E(\sigma).$$

Moreover if $f(\sigma \cap \sigma(R))$ is bounded then

$$f\left(R_\sigma \upharpoonright G_\sigma\right)E(\sigma) \in B(G).$$

PROOF. Let $y \in Dom(f(R)E(\sigma))$ then $E(\sigma)y \in G_\sigma \cap Dom(f(R))$ hence by (38), Lemma 1.7

$$f(R)E(\sigma)y = (f(R)_\sigma \upharpoonright G_\sigma)E(\sigma)y = f(R_\sigma \upharpoonright G_\sigma)E(\sigma)y.$$

So $f(R)E(\sigma) \subseteq f(R_\sigma \upharpoonright G_\sigma)E(\sigma)$. Next let $y \in Dom(f(R_\sigma \upharpoonright G_\sigma)E(\sigma))$, then $E(\sigma)y \in Dom(f(R_\sigma \upharpoonright G_\sigma))$, hence by Lemma 1.7 and (38)

$$f(R_\sigma \upharpoonright G_\sigma)(E(\sigma)y) = f(R)E(\sigma)E(\sigma)y = f(R)E(\sigma)y.$$

So $f(R_\sigma \upharpoonright G_\sigma)E(\sigma) \subseteq f(R)E(\sigma)$. Thus we obtain statement (1). Statement (2) follows by statement (1) and statement (3) of Lemma 1.7. □

2. Extension theorem for strong operator integral equalities

NOTATIONS 1.9. Let X be a locally compact space and μ a measure on X in the sense of the Bourbaki text [**INT**] see $III.7$, Definition 2, that is a continuous linear $\mathbb{C}-$functional on the $\mathbb{C}-$locally convex space $H(X)$ of all compactly supported complex continuous functions on X, with the direct limit topology (or inductive limit) of the spaces $H(X;K)$ with K running in the class of all compact subsets of X, where $H(X;K)$ is the space of all complex continuous functions $f : X \to \mathbb{C}$ such that supp$(f) \doteqdot \overline{\{x \in X \mid f(x) \neq 0\}} \subseteq K$ with the norm topology of uniform convergence [5]. In the work any measure μ on X in the sense of [**INT**] will be called complex Radon measure on X. For the definition of *$\mu-$integrable functions defined on X and with values in a $\mathbb{C}-$Banach space G* see $IV.23$. Definition 2 of [**INT**], while the *integral with respect to μ of a $\mu-$integrable function $f : X \to G$*, which will be denoted with $\int f(x)\,d\,\mu(x) \in G$, is defined in Definition 1, $III.33$ and Definition 1, $IV.33$ of [**INT**]. For the definition of the total variation $|\mu|$, and definition and properties of the upper integral $\int^* g\,d|\mu|(x)$ see Ch. $3-4$ of [**INT**]. We denote by $Comp(X)$ the class of the compact subsets of X and by $\mathfrak{F}_1(X;\mu)$ the seminormed space, with seminorm $\|\cdot\|_{\mathfrak{F}_1(X;\mu)}$, of all maps $F : X \to \mathbb{C}$ such that

$$\|F\|_{\mathfrak{F}_1(X;\mu)} \doteqdot \int^* |F(x)|\,d\,|\mu|(x) < \infty.$$

In this section it will be assumed, unless the contrary is stated, that X is a locally compact space and μ is a complex Radon measure over X. Let $B \subseteq X$ be a $\mu-$measurable set, then by $\mu - a.e.(B)$ we mean " almost everywhere in B with respect to the measure μ ". Let $f : X \to B(G)$ be a map $\mu-$integrable in the normed space $B(G)$ (Definition 2 Ch. IV, §3, $n°4$ of [**INT**]) then we convene to denote with the symbol

$$\oint f(x)\,d\,\mu(x) \in B(G)$$

[5] $H(X;K)$ is isometric to the Banach space of all continuous maps $g : K \to \mathbb{C}$ equal to 0 on ∂K, with the norm topology of uniform convergence

its integral in $B(G)$ (Definition 1 Ch. IV, §4, $n^{\circ}1$ of [**INT**]), which is uniquely determined by the following property for all $\phi \in B(G)^{*}$

$$\phi(\oint f(x)\, d\,\mu(x)) = \int \phi(f(x))\, d\,\mu(x).$$

For any scalar type spectral operator S in a complex Banach space G and for any Borelian map $f : U \supseteq \sigma(S) \to \mathbb{C}$ we assume that $f(S)$ is the closed operator defined in (35) and recall that we denote by $\widetilde{f}$ the $\mathbf{0}-$extension of f to $\mathbb{C}$, see Definition 1.2.

DEFINITION 1.10 ($E-$sequence). Let $E : \mathcal{B}_Y \to Pr(G)$ be a spectral measure in G on $\mathcal{B}_Y$ then we say that $\{\sigma_n\}_{n\in\mathbb{N}}$ is an $E-$*sequence* if there exists an $S \in \mathcal{B}_Y$ such that $E(S) = \mathbf{1}$ and

- $(\forall n \in \mathbb{N})(\sigma_n \in \mathcal{B}_Y)$;
- $(\forall n, m \in \mathbb{N})(n > m \Rightarrow \sigma_n \supseteq \sigma_m)$;
- $S \subseteq \bigcup_{n\in\mathbb{N}} \sigma_n$.

PROPOSITION 1.11. *Let $E : \mathcal{B}_Y \to Pr(G)$ be a countably additive spectral measure in G on a $\sigma-$ field $\mathcal{B}_Y$, and $\{\sigma_n\}_{n\in\mathbb{N}}$ an $E-$sequence. Then*

$$\lim_{n\to\infty} E(\sigma_n) = \mathbf{1} \quad \textit{in strong operator topology.} \tag{47}$$

PROOF. Let $S \in \mathcal{B}_Y$ of which in Definition 1.10 associated to the $E-$sequence $\{\sigma_n\}_{n\in\mathbb{N}}$. So $E(S) = \mathbf{1}$ and E is an order-preserving map, then $E(\bigcup_{n\in\mathbb{N}} \sigma_n) \geq E(S) = \mathbf{1}$. Since $\mathbf{1}$ is a maximal element in $\langle E(\mathcal{B}_Y), \geq\rangle$

$$E(\bigcup_{n\in\mathbb{N}} \sigma_n) = \mathbf{1}.$$

Let us define $\eta_1 \doteqdot \sigma_1$, and for all $n \geq 2$, $\eta_n \doteqdot \sigma_n \cap \complement\sigma_{n-1}$, so for all $n \in \mathbb{N}$, $\sigma_n = \bigcup_{k=1}^{n} \eta_k$, and for all $n \neq m \in \mathbb{N}$, $\eta_n \cap \eta_m = \emptyset$, finally $\bigcup_{n\in\mathbb{N}} \eta_n = \bigcup_{n\in\mathbb{N}} (\bigcup_{k=1}^{n} \eta_k) = \bigcup_{n\in\mathbb{N}} \sigma_n$. Therefore by the countable additivity of E with respect to the strong operator topology

$$\begin{aligned} E(\bigcup_{n\in\mathbb{N}} \sigma_n) &= E(\bigcup_{n\in\mathbb{N}} \eta_n) = \sum_{n=1}^{\infty} E(\eta_n) \\ &= \lim_{n\to\infty} \sum_{k=1}^{n} E(\eta_k) = \lim_{n\to\infty} E(\bigcup_{k=1}^{n} \eta_k) \\ &= \lim_{n\to\infty} E(\sigma_n). \end{aligned}$$

Here all limits are with respect to the strong operator topology, hence the statement. $\square$

DEFINITION 1.12 (Integration in the Strong Operator Topology). Let G_1, G_2 be two complex Banach spaces, and $f : X \to B(G_1, G_2)$. Then we say that *f is $\mu-$ integrable with respect to the strong operator topology* if

(1) for all $v \in G_1$ the map $X \ni x \mapsto f(x)v \in G_2$ is $\mu-$integrable;
(2) if we set

$$F : G_1 \ni v \mapsto \int f(x)(v)\, d\,\mu(x) \in G_2$$

then $F \in B(G_1, G_2)$.

In such a case we set $\int f(x)\,d\,\mu(x) \doteqdot F$, in other words the integral $\int f(x)\,d\,\mu(x)$ of f with respect to the measure μ and the strong operator topology is a bounded linear operator from G_1 to G_2 such that for all $v \in G_1$

$$\left(\int f(x)\,d\,\mu(x)\right)(v) = \int f(x)(v)\,d\,\mu(x).$$

We shall need the following version of the Minkowski inequality

PROPOSITION 1.13. *Let G_1, G_2 be two complex Banach spaces, and a map $f : X \to B(G_1, G_2)$ such that*

(1) $(\forall v \in G_1)(\forall \phi \in G_2^*)$ *the complex map* $X \ni x \mapsto \phi(f(x)v) \in \mathbb{C}$ *is* $\mu-$*measurable;*
(2) *for all $v \in G_1, K \in Comp(X)$ there is $H \subset G_2$ such that H is countable and $f(x)v \in \overline{H}\ \ \mu - a.e.(K)$;*
(3) $(X \ni x \mapsto \|f(x)\|_{B(G_1,G_2)}) \in \mathfrak{F}_1(X;\mu)$,

Then f is $\mu-$integrable with respect to the strong operator topology and we have

$$\left\|\int f(x)\,d\,\mu(x)\right\|_{B(G_1,G_2)} \leq \int^* \|f(x)\|_{B(G_1,G_2)}\,d\,|\mu|(x).$$

PROOF. By hypothesis (3) we have for all $v \in G_1$

$$\int^* \|f(x)v\|_{G_2}\,d\,|\mu|(x) \leq \|v\|_{G_1} \int^* \|f(x)\|_{B(G_1,G_2)}\,d\,|\mu|(x) < \infty. \tag{48}$$

By hypothesis $(1-2)$ and Corollary 1, $IV.70$ of [**INT**], we have for all $v \in G_1$ that the map $X \mapsto f(x)v \in G_2$ is $\mu-$measurable. Therefore by (48) and by Theorem 5, $IV.71$ of [**INT**] we deduce for all $v \in G_1$ that $X \mapsto f(x)v \in G_2$ is $\mu-$integrable. So in particular by Definition 1, $IV.33$ of [**INT**] for all $v \in G_1$ there is $\int f(x)v\,d\,\mu(x) \in G_2$ while by Proposition 2, $IV.35$ of [**INT**] and the (48) we obtain for all $v \in G_1$

$$\left\|\int f(x)v\,d\,\mu(x)\right\|_{G_2} \leq \|v\|_{G_1} \int^* \|f(x)\|_{B(G_1,G_2)}\,d\,|\mu|(x)$$

Hence the statement follows. □

REMARK 1.14. As it follows by the above proof Proposition 1.13 is also valid if we replace the hypotheses $(1-2)$ with the following one

(1') $\quad \forall v \in G_1$ the map $X \ni x \mapsto f(x)v \in G_2$ is $\mu-$measurable.

LEMMA 1.15. *Let X, Y, Z be three normed spaces over the same field $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$, $R : Dom(R) \subseteq Y \to Z$ a possibly unbounded closed linear operator and $A \in B(X, Y)$. Then $R \circ A : \mathsf{D} \to Z$ is a closed operator, where $\mathsf{D} \doteqdot Dom(R \circ A)$*

PROOF. Let $\{x_n\}_{n\in\mathbb{N}} \subset \mathsf{D} \doteqdot \{x \in X \mid A(x) \in Dom(R)\}$, and $(x, z) \in X \times Z$ such that $x = \lim_{n\to\infty} x_n$, and $z = \lim_{n\to\infty} R \circ A(x_n)$. A being continuous we have $A(x) = \lim_{n\to\infty} A(x_n)$, but $z = \lim_{n\to\infty} R(Ax_n)$, and R is closed, so $z = R(A(x)) \doteqdot R \circ A(x)$, hence $(x, z) \in Graph(R \circ A)$, which is just the statement. □

LEMMA 1.16. *Let X be a normed space and Y a Banach space over the same field $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$, finally $U : \mathsf{D} \subseteq X \to Y$ be a linear operator. If U is continuous and closed then D is closed.*

PROOF. Let $\{x_n\}_{n\in\mathbb{N}} \subset \mathsf{D}$ and $x \in X$ such that $x = \lim_{n\to\infty} x_n$. So by the continuity of U we have for all $n, m \in \mathbb{N}$ that $\|U(x_n) - U(x_m)\| = \|U(x_n - x_m)\| \leq \|U\|\|x_n - x_m\|$, hence $\lim_{(n,m)\in\mathbb{N}^2} \|U(x_n) - U(x_m)\| = 0$, thus Y being a Banach space we have that there is $y \in Y$ such that $y = \lim_{n\to\infty} U(x_n)$. But U is closed, therefore $y = U(x)$, so $x \in \mathsf{D}$, which is the statement. $\square$

THEOREM 1.17. *Let R be a possibly unbounded scalar type spectral operator in G, $\sigma(R)$ its spectrum and E its resolution of the identity. Let the map $X \ni x \mapsto f_x \in Bor(\sigma(R))$ be such that for all $x \in X$, $\widetilde{f_x} \in \mathfrak{L}_E^{\infty}(\sigma(R))$ where $X \ni x \mapsto f_x(R) \in B(G)$ is $\mu-$integrable with respect to the strong operator topology.*

Then

1. *for all $\sigma \in \mathcal{B}(\mathbb{C})$ the map $X \ni x \mapsto f_x(R_\sigma \upharpoonright G_\sigma) \in B(G_\sigma)$ is $\mu-$integrable with respect to the strong operator topology and*
$$\left\|\int f_x(R_\sigma \upharpoonright G_\sigma)\, d\,\mu(x)\right\|_{B(G_\sigma)} \leq \left\|\int f_x(R)\, d\,\mu(x)\right\|_{B(G)}.$$
2. *If $g, h \in Bor(\sigma(R))$, $\{\sigma_n\}_{n\in\mathbb{N}}$ is an $E-$sequence, and for all $n \in \mathbb{N}$*

$$g(R_{\sigma_n} \upharpoonright G_{\sigma_n}) \int f_x(R_{\sigma_n} \upharpoonright G_{\sigma_n})\, d\,\mu(x) \subseteq h(R_{\sigma_n} \upharpoonright G_{\sigma_n}). \tag{49}$$

then

$$g(R) \int f_x(R)\, d\,\mu(x) \upharpoonright \Theta = h(R) \upharpoonright \Theta, \tag{50}$$

where $\Theta \doteqdot Dom\left(g(R) \int f_x(R)\, d\,\mu(x)\right) \cap Dom(h(R))$ and all the integrals are with respect to the strong operator topologies.

Notice that $g(R)$ is possibly an **unbounded** operator in G.

PROOF. Let $\sigma \in \mathcal{B}(\mathbb{C})$ then by (43)

$$(\forall \sigma \in \mathcal{B}(\mathbb{C}))(\sigma(R_\sigma \upharpoonright G_\sigma) \subseteq \overline{\sigma} \cap \sigma(R) \subseteq \sigma(R)).$$

which implies that all the following operator functions $g(R_\sigma \upharpoonright G_\sigma)$, $h(R_\sigma \upharpoonright G_\sigma)$ and for all $x \in X$ the $f_x(R_\sigma \upharpoonright G_\sigma)$, are well defined. By the fact that $\{\delta \in \mathcal{B}(\mathbb{C}) \mid E(\delta) = \mathbf{1}\} \subseteq \{\delta \in \mathcal{B}(\mathbb{C}) \mid \widetilde{E}_\sigma(\delta) = \mathbf{1}_\sigma\}$ which follows by statement (1) of Lemma 1.7, we deduce for all $x \in X$

$$\|\widetilde{f}_x\|_\infty^{\widetilde{E}_\sigma} \leq \|\widetilde{f}_x\|_\infty^E = \|\widetilde{f}_x \chi_{\sigma(R)}\|_\infty^E < \infty,$$

where the last equality came by $\widetilde{f}_x \chi_{\sigma(R)} = \widetilde{f}_x$, while the boundedness by the hypothesis $\widetilde{f}_x \in \mathfrak{L}_E^\infty(\sigma(R))$. Thus $\widetilde{f}_x \in \mathfrak{L}_{\widetilde{E}_\sigma}^\infty(\mathbb{C})$ hence by statement (c) of Theorem $18.2.11$. of [**DS**] applied to the scalar type spectral operator $R_\sigma \upharpoonright G_\sigma$

$$(\forall \sigma \in \mathcal{B}(\mathbb{C}))(f_x(R_\sigma \upharpoonright G_\sigma) \in B(G_\sigma)). \tag{51}$$

A more direct way for obtaining (51) is to use statement (2) of Lemma 1.7 and the fact that $\widetilde{f}_x \in \mathfrak{L}_E^\infty(\sigma(R))$ implies $f_x(R) \in B(G)$. For all $\sigma \in \mathcal{B}(\mathbb{C})$ we claim that $X \ni x \mapsto f_x(R_\sigma \upharpoonright G_\sigma) \in B(G_\sigma)$ is μ-integrable with respect to the strong operator topology. By Lemma 1.7 we have for all $\sigma \in \mathcal{B}(\mathbb{C})$ and for all $v \in G_\sigma$

$$\int^* \|f_x(R_\sigma \upharpoonright G_\sigma)v\|_{G_\sigma}\, d\,|\mu|(x) = \int^* \|f_x(R)v\|_G\, d\,|\mu|(x) < \infty. \tag{52}$$

Here the boundedness comes by Theorem 5, $IV.71$ of [**INT**] applied to the $\mu-$integrable map $X \ni x \mapsto f_x(R)v \in G$. By Corollary 1, $IV.70$ and Theorem 5, $IV.71$ of [**INT**]

applied, for any $v \in G$, to the $\mu-$integrable map $X \ni x \mapsto f_x(R)v \in G$, we have for all $v \in G, K \in Comp(X)$ there is $H^v \subseteq G$ countable such that $(f_x(R)v \in \overline{H^v}$, $\mu - a.e.(K))$. But by statement (g) of Theorem $18.2.11.$ of [**DS**] and $f_x(R) \in B(G)$, we have for all $\sigma \in \mathcal{B}(\mathbb{C})$, $[f_x(R), E(\sigma)] = \mathbf{0}$, hence by the previous equation and by the fact that $E(\sigma) \in B(G)$, so it is continuous, we obtain for all $\sigma \in \mathcal{B}(\mathbb{C}), v \in G, K \in Comp(X)$

$$(\exists\, H^v \subseteq G \text{ countable })(f_x(R)E(\sigma)v = E(\sigma)f_x(R)v \in \overline{H^v_\sigma},\ \mu - a.e.(K)).$$

Here $H^v_\sigma \doteqdot E(\sigma)H^v$. Therefore by Lemma 1.7 we state that for all $\sigma \in \mathcal{B}(\mathbb{C}), v \in G_\sigma, K \in Comp(X)$

$$(\exists\, H^v_\sigma \subset G_\sigma \text{ countable })(f_x(R_\sigma \restriction G_\sigma)v \in \overline{H^v_\sigma} \subseteq G_\sigma,\ \mu - a.e.(K)). \tag{53}$$

That $\overline{H^v_\sigma} \subseteq G_\sigma$ follows by the fact that G_σ is closed in G. Therefore we can consider the closure $\overline{H^v_\sigma}$ as the closure in the Banach space G_σ. By the Hahn-Banach Theorem, see Corollary 3, $II.23$ of the [**TVS**], for all $\sigma \in \mathcal{B}(\mathbb{C})$

$$\{\phi \restriction G_\sigma \mid \phi \in G^*\} = (G_\sigma)^*. \tag{54}$$

Moreover by Corollary 1, $IV.70$ and Theorem 5, $IV.71$ of [**INT**] applied, for any $v \in G$, to the $\mu-$integrable map $X \ni x \mapsto f_x(R)E(\sigma)v \in G$, we have for all $\phi \in G^*$

$$X \ni x \mapsto \phi(f_x(R)E(\sigma)v) \in \mathbb{C} \text{ is } \mu\text{−measurable.}$$

Thus by Lemma 1.7 we have for all $\sigma \in \mathcal{B}(\mathbb{C}), v \in G_\sigma, \phi \in G^*$

$$X \ni x \mapsto \phi(f_x(R_\sigma \restriction G_\sigma)v) \in \mathbb{C} \text{ is } \mu\text{−measurable.}$$

Hence by (54) we can state for all $\sigma \in \mathcal{B}(\mathbb{C}), v \in G_\sigma$ that

$$(\forall \phi_\sigma \in (G_\sigma)^*)(X \ni x \mapsto \phi_\sigma(f_x(R_\sigma \restriction G_\sigma)v) \in \mathbb{C} \text{ is } \mu\text{−measurable.}) \tag{55}$$

Now by collecting (55), (52) and (53), where the closure $\overline{H^v_\sigma}$ is to be intended how the closure in the Banach space G_σ, we can apply Corollary 1, $IV.70$ and Theorem 5, $IV.71$ of [**INT**] to the map $X \ni x \mapsto f_x(R_\sigma \restriction G_\sigma)v \in G_\sigma$, in order to state that

$$(\forall \sigma \in \mathcal{B}(\mathbb{C}))(\forall v \in G_\sigma)(X \ni x \mapsto f_x(R_\sigma \restriction G_\sigma)v \in G_\sigma \text{ is } \mu\text{−integrable.}) \tag{56}$$

This means in particular that there exists its integral, so for all $\sigma \in \mathcal{B}(\mathbb{C}), v \in G_\sigma$

$$\begin{aligned}\left\|\int f_x(R_\sigma \restriction G_\sigma)v\, d\,\mu(x)\right\|_{G_\sigma} &= \left\|\int f_x(R)v\, d\,\mu(x)\right\|_G && \text{by Lemma 1.7}\\ &\leq \left\|\int f_x(R)\, d\,\mu(x)\right\|_{B(G)} \|v\|_{G_\sigma}. && (57)\end{aligned}$$

Here the inequality follows by the hypothesis that $X \ni x \mapsto f_x(R) \in B(G)$ is $\mu-$integrable in the strong operator topology. Therefore by Definition 1.12 and (51), (56), (57) we can conclude that

(58)

$$\begin{cases}(\forall \sigma \in \mathcal{B}(\mathbb{C}))(X \ni x \mapsto f_x(R_\sigma \restriction G_\sigma) \in B(G_\sigma) \text{ is } \mu\text{−integr. in strong operator topology)}\\ \left\|\int f_x(R_\sigma \restriction G_\sigma)\, d\,\mu(x)\right\|_{B(G_\sigma)} \leq \left\|\int f_x(R)\, d\,\mu(x)\right\|_{B(G)}.\end{cases}$$

Which is the claim we wanted to show, then statement (1) follows. Statement (1) proves that the assumption (49) is well set, so we are able to start the proof of the statement (2).

For all $y \in \Theta$

$$
\begin{aligned}
&= g(R)\int f_x(R)\,d\,\mu(x)y \\
&= \lim_{n\in\mathbb{N}} E(\sigma_n)g(R)\int f_x(R)\,d\,\mu(x)y && \text{by (47)} \\
&= \lim_{n\in\mathbb{N}} g(R)E(\sigma_n)\int f_x(R)\,d\,\mu(x)y && \text{by } (g) \text{ of Theorem } 18.2.11 \text{ of [\textbf{DS}]} \\
&= \lim_{n\in\mathbb{N}} g(R)E(\sigma_n)\int f_x(R)y\,d\,\mu(x) && \text{by Definition 1.12} \\
&= \lim_{n\in\mathbb{N}} g(R)E(\sigma_n)\int E(\sigma_n)f_x(R)y\,d\,\mu(x) && \text{by Theorem } 1, IV.35 \text{ of [\textbf{INT}]} \\
&= \lim_{n\in\mathbb{N}} g(R)E(\sigma_n)\int f_x(R)E(\sigma_n)y\,d\,\mu(x) && \text{by } (g) \text{ of Theorem } 18.2.11 \text{ of [\textbf{DS}]} \\
&= \lim_{n\in\mathbb{N}} g(R_{\sigma_n}\upharpoonright G_{\sigma_n})\int f_x(R_{\sigma_n}\upharpoonright G_{\sigma_n})E(\sigma_n)y\,d\,\mu(x) && \text{by \textbf{Lemma 1.7}} \\
&= \lim_{n\in\mathbb{N}} g(R_{\sigma_n}\upharpoonright G_{\sigma_n})\int f_x(R_{\sigma_n}\upharpoonright G_{\sigma_n})\,d\,\mu(x)E(\sigma_n)y && \text{by statement } (1) \text{ and Definition 1.12} \\
&= \lim_{n\in\mathbb{N}} h(R_{\sigma_n}\upharpoonright G_{\sigma_n})E(\sigma_n)y && \text{by hypothesis (49)} \\
&= \lim_{n\in\mathbb{N}} h(R)E(\sigma_n)y && \text{by Lemma 1.7} \\
&= \lim_{n\in\mathbb{N}} E(\sigma_n)h(R)y && \text{by } (g) \text{ of Theorem } 18.2.11 \text{ of [\textbf{DS}]} \\
&h(R)y && \text{by (47).}
\end{aligned}
$$

Therefore

$$g(R)\int f_x(R)\,d\,\mu(x)\upharpoonright\Theta = h(R)\upharpoonright\Theta.$$

□

THEOREM 1.18 (**Strong Extension Theorem**). *Let X be a locally compact space, μ a complex Radon measure on X, R be a possibly unbounded scalar type spectral operator in G, $\sigma(R)$ its spectrum and E its resolution of the identity. Let the map $X \ni x \mapsto f_x \in Bor(\sigma(R))$ be such that for all $x \in X$, $\widetilde{f_x} \in \mathfrak{L}_E^\infty(\sigma(R))$, where the map $X \ni x \mapsto f_x(R) \in B(G)$ be $\mu-$integrable with respect to the strong operator topology. Finally let $g, h \in Bor(\sigma(R))$ and $\widetilde{h} \in \mathfrak{L}_E^\infty(\sigma(R))$.*

If $\{\sigma_n\}_{n\in\mathbb{N}}$ is an $E-$sequence and for all $n \in \mathbb{N}$

$$g(R_{\sigma_n}\upharpoonright G_{\sigma_n})\int f_x(R_{\sigma_n}\upharpoonright G_{\sigma_n})\,d\,\mu(x) \subseteq h(R_{\sigma_n}\upharpoonright G_{\sigma_n}) \tag{59}$$

then $h(R) \in B(G)$ and

$$g(R)\int f_x(R)\,d\,\mu(x) = h(R).$$

Here all the integrals are with respect to the strong operator topologies.

Notice that $g(R)$ is possibly an **unbounded** operator on G.

PROOF. $h(R) \in B(G)$ by Theorem 18.2.11. of [**DS**] and the hypothesis $\widetilde{h} \in \mathfrak{L}_E^\infty(\sigma(R))$, so by (50)

$$g(R) \int f_x(R)\, d\,\mu(x) \subseteq h(R). \tag{60}$$

Let us set

$$(\forall n \in \mathbb{N})(\delta_n \doteqdot \overset{-1}{|g|}([0,n])). \tag{61}$$

We claim that

$$\begin{cases} \bigcup_{n\in\mathbb{N}} \delta_n = \sigma(R) \\ n \geq m \Rightarrow \delta_n \supseteq \delta_m \\ (\forall n \in \mathbb{N})(g(\delta_n) \text{ is bounded. }) \end{cases} \tag{62}$$

In addition being $|g| \in Bor(\sigma(R))$ we have $\delta_n \in \mathcal{B}(\mathbb{C})$ for all $n \in \mathbb{N}$, so $\{\delta_n\}_{n\in\mathbb{N}}$ is an $E-$sequence, hence by (47)

$$\lim_{n\in\mathbb{N}} E(\delta_n) = \mathbf{1} \tag{63}$$

with respect to the strong operator topology on $B(G)$. Indeed the first equality follows since $\bigcup_{n\in\mathbb{N}} \delta_n \doteq \bigcup_{n\in\mathbb{N}} \overset{-1}{|g|}([0,n]) = \overset{-1}{|g|}\left(\bigcup_{n\in\mathbb{N}}[0,n]\right) = \overset{-1}{|g|}(\mathbb{R}^+) = Dom(g) \doteqdot \sigma(R)$, the second by the fact that $\overset{-1}{|g|}$ preserves the inclusion, the third since $|g|(\delta_n) \subseteq [0,n]$. Hence our claim. By the third statement of (62), $\delta_n \in \mathcal{B}(\mathbb{C})$ and statement 3 of Lemma 1.7

$$(\forall n \in \mathbb{N})(E(\delta_n)G \subseteq Dom(g(R))). \tag{64}$$

$f_x(R)E(\delta_n) = E(\delta_n)f_x(R)$, by statement (g) of Theorem 18.2.11 of [**DS**], so for all $v \in G$

$$\begin{aligned} \int f_x(R)\, d\,\mu(x) E(\delta_n)v &\doteq \int f_x(R)E(\delta_n)v\, d\,\mu(x) \\ &= \int E(\delta_n)f_x(R)v\, d\,\mu(x) = E(\delta_n)\int f_x(R)v\, d\,\mu(x), \end{aligned}$$

where the last equality follows by applying Theorem 1, $IV.35$. of [**INT**]. Hence for all $n \in \mathbb{N}$

$$\int f_x(R)\, d\,\mu(x) E(\delta_n)G \subseteq E(\delta_n)G \subseteq Dom(g(R)),$$

where the last inclusion is by (64). Therefore

$$(\forall n \in \mathbb{N})(\forall v \in G)\left(E(\delta_n)v \in Dom\left(g(R)\int f_x(R)\, d\,\mu(x)\right)\right).$$

Hence by (63)

$$\mathbf{D} \doteqdot Dom\left(g(R)\int f_x(R)\, d\,\mu(x)\right) \text{ is dense in } G. \tag{65}$$

Next $\int f_x(R)\, d\,\mu(x) \in B(G)$ and $g(R)$ is closed by Theorem 18.2.11. of [**DS**], so by Lemma 1.15

$$g(R)\int f_x(R)\, d\,\mu(x) \text{ is closed.} \tag{66}$$

Moreover $h(R) \in B(G)$ hence by (60)

$$g(R)\int f_x(R)\, d\,\mu(x) \in B(\mathbf{D}, G). \tag{67}$$

(66), (67) and Lemma 1.16 allow us to state that $\mathbf{D}$ is closed in G, thus by (65) we have

$$\mathbf{D} = G.$$

Therefore by (60) the statement follows. □

Now we shall prove a corollary of the previous result in which conditions are given ensuring the strong operator integrability of the map $f_x(R)$.

COROLLARY 1.19. *Let R be a possibly unbounded scalar type spectral operator in G. Let $\{\sigma_n\}_{n\in\mathbb{N}}$ be an $E-$sequence and for all $x \in X$, $f_x \in Bor(\sigma(R))$ such that*

$$(X \ni x \mapsto \|\widetilde{f_x}\|_\infty^E) \in \mathfrak{F}_1(X;\mu)$$

and $X \ni x \mapsto f_x(R) \in B(G)$ satisfies the conditions $(1-2)$ of Proposition 1.13, (respectively for all $v \in G$ the map $X \ni x \mapsto f_x(R)v \in G$ is $\mu-$measurable). Finally let $g, h \in Bor(\sigma(R))$. If we assume that for all $n \in \mathbb{N}$ holds (59) and that $\widetilde{h} \in \mathfrak{L}_E^\infty(\sigma(R))$, then the same conclusions of Thm. 1.18 hold.

PROOF. By statement (c) of Theorem 18.2.11 of [**DS**] and Proposition 1.13, (respectively Remark 1.14) the map $X \ni x \mapsto f_x(R) \in B(G)$ is $\mu-$integrable with respect to the strong operator topology and

$$\left\| \int f_x(R)\, d\,\mu(x) \right\|_{B(G)} \leq 4M \int^* \|\widetilde{f_x}\|_\infty^E \, d\,|\mu|(x)$$

Here $M \doteqdot \sup_{\sigma\in\mathcal{B}(\mathbb{C})} \|E(\sigma)\|_{B(G)}$. Therefore the statement follows by Theorem 1.18. □

3. Generalization of the Newton-Leibnitz formula

The main result of this section is Theorem 1.25 which generalizes the Newton-Leibnitz formula to the case of unbounded scalar type spectral operators in G. For proving Theorem 1.25 we need two preliminary results, the first is Theorem 1.21, concerning the Newton-Leibnitz formula for any bounded scalar type spectral operator on G and any analytic map on an open neighbourhood of its spectrum. The second, Theorem 1.23, concerns strong operator continuity, and under additional conditions also differentiability, for operator maps of the type $K \ni t \mapsto S(tR) \in B(G)$, where K is an open interval of $\mathbb{R}$, $S(tR)$ arises by the Borel functional calculus for the unbounded scalar type spectral operator R in G and S is any analytic map on an open neighbourhood U of $\sigma(R)$ such that $K \cdot U \subseteq U$. Let Z be a non empty set, Y a $\mathbb{K}-$linear space ($\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$), $U \subseteq Y$, $K \subseteq \mathbb{K}$ such that $K \cdot U \subseteq U$ and $F : U \to Z$. Then we set $F_t : U \to Z$ such that $F_t(\lambda) \doteqdot F(t\lambda)$, for all $t \in K$ and $\lambda \in U$. If F, G are two $\mathbb{C}-$Banach spaces, $A \subseteq F$ open and $f : A \subseteq F \to G$ a map, we convene to denote the real Banach spaces $F_\mathbb{R}$ and $G_\mathbb{R}$ associated to F and G again by F and G respectively, and with the symbol f the map $f^\mathbb{R} : A \subseteq F_\mathbb{R} \to G_\mathbb{R}$.

LEMMA 1.20. *Let $\langle Y, d\rangle$ be a metric space, U an open of Y and σ a compact such that $\sigma \subseteq U$. Then there is $Q > 0$*

$$K \doteqdot \overline{\bigcup_{\{y\in\sigma\}} \overline{B}_Q(y)} \subseteq U, \tag{68}$$

moreover if σ is of finite diameter then K is of finite diameter.

PROOF. By Remark §2.2., Ch. 9 of [**GT**] we deduce

$$P \doteqdot dist(\sigma, \complement U) \neq 0,$$

where $dist(A, B) \doteqdot \inf_{\{x\in A, y\in B\}} d(x, y)$, for all $A, B \subseteq Y$. Set

$$Q \doteqdot \frac{P}{2}$$

then for all $y \in \sigma, x \in \overline{B}_Q(y), z \in \complement U$ we have

$$d(x, z) \geq d(z, y) - d(y, x) \geq \frac{P}{2} \neq 0. \tag{69}$$

Thus by applying Proposition 2, §2.2., Ch. 9 of [**GT**] $\overline{B}_Q(y) \cap \complement U = \emptyset$, i.e. $\overline{B}_Q(y) \subseteq U$, then

$$A \doteqdot \bigcup_{\{y\in\sigma\}} \overline{B}_Q(y) \subseteq U.$$

Moreover by Proposition 3, §2.2., Ch. 9 of [**GT**] the map $x \mapsto d(x, \complement U)$ is continuous on Y, hence by (69) for all $x \in \overline{A}$

$$d(x, \complement U) = \lim_{n\in\mathbb{N}} d(x_n, \complement U) \geq \frac{P}{2} \neq 0,$$

for all $\{x_n\}_{n\in\mathbb{N}} \subset A$ such that $x = \lim_{n\in\mathbb{N}} x_n$. Therefore by Proposition 2, §2.2., Ch. 9 of [**GT**] (68) follows. Let $B \subset Y$ be of finite diameter then by the continuity of the map $d : Y \times Y \to \mathbb{R}^+$ it is of finite diameter also $\overline{B}$. Indeed let $diam(B) \doteqdot \sup_{\{x,y\in B\}} d(x, y)$, if by absurdum $\sup_{\{x,y\in\overline{B}\}} d(x, y) = \infty$ then

$$(\exists\, x_0, y_0 \in \overline{B})(d(x_0, y_0) > diam(B) + 1). \tag{70}$$

Let $\{(x_\alpha, y_\alpha)\}_{\alpha\in D} \subset B \times B$ be a net such that $\lim_{\alpha\in D}(x_\alpha, y_\alpha) = (x_0, y_0)$ limit in $\langle Y, d\rangle \times \langle Y, d\rangle$. Thus by the continuity of d

$$d(x_0, y_0) = \lim_{\alpha\in D} d(x_\alpha, y_\alpha) \leq diam(B)$$

which contradicts (70), so $\sup_{\{x,y\in\overline{B}\}} d(x, y) < \infty$. Therefore if A is of finite diameter it is so K. Let $z_1, z_2 \in A$ then there exist $y_1, y_2 \in \sigma$ such that $z_k \in \overline{B}_Q(y_k)$, for $k \in \{1, 2\}$. Then

$$d(z_1, z_2) \leq d(z_1, y_1) + d(y_1, y_2) + d(y_2, z_2) \leq 2Q + diam(\sigma) < \infty,$$

where $diam(\sigma) \doteqdot \sup_{\{x,y\in\sigma\}} d(x, y)$. Hence A is of finite diameter. □

THEOREM 1.21. *Let $T \in B(G)$ be a scalar type spectral operator, $\sigma(T)$ its spectrum. Assume that $0 < L \leq \infty$, U is an open neighbourhood of $\sigma(T)$ such that $]-L, L[\cdot U \subseteq U$ and $F : U \to \mathbb{C}$ is an analytic map. Then for all $t \in]-L, L[$*

(1)
$$F(tT) = F_t(T); \tag{71}$$

(2)
$$\frac{d\,F(tT)}{d\,t} = T\frac{d\,F}{d\,\lambda}(tT); \tag{72}$$

(3) *for all $u_1, u_2 \in]-L, L[$*
$$T \oint_{u_1}^{u_2} \frac{d\,F}{d\,\lambda}(tT)d\,t = F(u_2T) - F(u_1T). \tag{73}$$

Here $F_t(T)$, (respectively $\frac{dF}{d\lambda}(tT)$ and $F(tT)$) are the operators arising by the Borelian functional calculus of the operator T (respectively tT) for all $t \in]-L, L[$.

PROOF. T is a bounded operator on G so $\sigma(T)$ is compact. Let us denote by $\langle \mathcal{C}(\sigma(T)), \|\cdot\|_{\sup}\rangle$ the Banach algebra of all continuous complex valued maps defined on $\sigma(T)$ with the norm $\|g\|_{\sup} \doteqdot \sup_{\lambda\in\sigma(T)} |g(\lambda)|$. Set

$$\begin{cases} \widetilde{\mathcal{C}}(\sigma(T)) \doteqdot \left\{ f : \mathbb{C} \to \mathbb{C} \mid f \upharpoonright \sigma(T) \in \mathcal{C}(\sigma(T)),\ f \upharpoonright \complement\sigma(T) = \mathbf{0} \right\}, \\ J : \mathcal{C}(\sigma(T)) \ni g \mapsto \widetilde{g} \in \widetilde{\mathcal{C}}(\sigma(T)). \end{cases} \tag{74}$$

Notice that $\widetilde{\mathcal{C}}(\sigma(T))$ is an algebra moreover J is a surjective morphism of algebras and $\sup_{\lambda\in\mathbb{C}} |J(g)(\lambda)| = \|g\|_{\sup}$ for all $g \in \mathcal{C}(\sigma(T))$ furthermore $J(g) \in Bor(\mathbb{C})$ since $g \in Bor(\sigma(T))$ and $\sigma(T) \in \mathcal{B}(\mathbb{C})$. Hence $\widetilde{\mathcal{C}}(\sigma(T))$ is a subalgebra of $\mathbf{TM}$, moreover J is an isometry between $\langle \mathcal{C}(\sigma(T)), \|\cdot\|_{\sup}\rangle$ and $\left\langle \widetilde{\mathcal{C}}(\sigma(T)), \|\cdot\|_{\sup}\right\rangle$. Thus $\left\langle \widetilde{\mathcal{C}}(\sigma(T)), \|\cdot\|_{\sup}\right\rangle$ is a Banach subalgebra of the Banach algebra $\langle \mathbf{TM}, \|\cdot\|_{\sup}\rangle$ and J is an isometric isomorphism of algebras. Therefore by denoting with E the resolution of the identity of T, by (30) we have that $\mathbf{I}_{\mathbb{C}}^{E} \circ J$ is a unital [6] morphism of algebras such that $\mathbf{I}_{\mathbb{C}}^{E} \circ J \in B(\langle \mathcal{C}(\sigma(T)), \|\cdot\|_{\sup}\rangle, B(G))$. In the sequel we convene to denote for brevity with the symbol $\mathbf{I}_{\mathbb{C}}^{E}$ the operator $\mathbf{I}_{\mathbb{C}}^{E} \circ J$ so

$$\begin{cases} \mathbf{I}_{\mathbb{C}}^{E} \in B(\langle \mathcal{C}(\sigma(T)), \|\cdot\|_{\sup}\rangle, B(G)), \\ \mathbf{I}_{\mathbb{C}}^{E} \text{ is a unital morphism of algebras} \\ (\forall g \in \mathcal{C}(\sigma(T)))(g(T) = \mathbf{I}_{\mathbb{C}}^{E}(g)). \end{cases} \tag{75}$$

In particular $\mathbf{I}_{\mathbb{C}}^{E}$ is Fréchet differentiable with constant differential map equal to $\mathbf{I}_{\mathbb{C}}^{E}$. In the sequel we shall denote with **0** the zero element of the Banach space $\langle \mathcal{C}(\sigma(T)), \|\cdot\|_{\sup}\rangle$. Let $t \in]-L, L[-\{0\}$, and $\imath_t \doteqdot t \cdot \imath$, where $\imath : \sigma(T) \ni \lambda \mapsto \lambda$. So $\imath_t(T) = \mathbf{I}_{\mathbb{C}}^{E}(t\cdot\imath) = t\mathbf{I}_{\mathbb{C}}^{E}(\imath) = tT$. Hence by the general spectral mapping theorem 18.2.21. of [**DS**] applied to the map $\imath_t$, the fact that $\sigma(T)$ is closed and the product by no zero scalars in $\mathbb{C}$ is a homeomorphism, we deduce that tT is a scalar type spectral operator and $E_t : \mathcal{B}(\mathbb{C}) \ni \delta \mapsto E(t^{-1}\delta)$ its resolution of the identity. Finally

$$(\forall t \in]-L, L[)(\sigma(tT) = t\sigma(T) \subseteq U),$$

the inclusion is by hypothesis. So $F(tT)$ arising by the Borel functional calculus of the operator tT is well defined and by (75)

$$\begin{aligned} F(tT) = \mathbf{I}_{\mathbb{C}}^{E_t}(F \upharpoonright \sigma(tT)) &\doteq \mathbf{I}_{\mathbb{C}}^{E\circ\imath_{t^{-1}}}(F \upharpoonright \sigma(tT)) \\ &= \mathbf{I}_{\mathbb{C}}^{E}(F \circ \imath_t) \\ &= \mathbf{I}_{\mathbb{C}}^{E}(F_t \upharpoonright \sigma(T)) = F_t(T). \end{aligned} \tag{76}$$

Thus (71). Set

$$\Delta :]-L, L[\ni t \mapsto (F \circ \imath_t) \in \langle \mathcal{C}(\sigma(T)), \|\cdot\|_{\sup}\rangle,$$

by the third equality in (76)

$$(\forall t \in]-L, L[)(F(tT) = \mathbf{I}_{\mathbb{C}}^{E} \circ \Delta(t)). \tag{77}$$

[6] indeed by setting $\mathbf{1} : \mathbb{C} \ni \lambda \mapsto 1 \in \mathbb{C}$ the unity element in $\mathbf{TM}$ then $\mathbf{I}_{\mathbb{C}}^{E} \circ J(\mathbf{1} \upharpoonright \sigma(T)) = \mathbf{I}_{\mathbb{C}}^{E}(\mathbf{1} \cdot \chi_{\sigma(T)}) = \mathbf{I}_{\mathbb{C}}^{E}(\mathbf{1})\mathbf{I}_{\mathbb{C}}^{E}(\chi_{\sigma(T)}) = \mathbf{1}$.

We claim that Δ is derivable (i.e. Fréchet differentiable) and for all $t \in]-L, L[$

$$\frac{d\,\Delta}{dt}(t) = \imath \cdot \left(\frac{d\,F}{d\lambda}\right)_t \upharpoonright \sigma(T). \tag{78}$$

Set

$$\begin{cases} \mathcal{C}_U(\sigma(T)) \doteqdot \{f \in \mathcal{C}(\sigma(T)) \mid f(\sigma(T)) \subseteq U\}, \\ \zeta :]-L, L[\ni t \mapsto \imath_t \in \mathcal{C}_U(\sigma(T)) \subset \langle \mathcal{C}(\sigma(T)), \|\cdot\|_{\sup}\rangle \\ \Upsilon : \mathcal{C}_U(\sigma(T)) \ni f \mapsto F \circ f \in \langle \mathcal{C}(\sigma(T)), \|\cdot\|_{\sup}\rangle \,. \end{cases}$$

Notice

$$\Delta = \Upsilon \circ \zeta, \tag{79}$$

moreover ζ is Fréchet differentiable and for all $t \in]-L, L[$

$$\frac{d\,\zeta}{dt}(t) = \imath. \tag{80}$$

Next for all $f \in \mathcal{C}_U(\sigma(T))$ by Lemma 1.20 applied to the compact $f(\sigma(T))$, there is $Q_f > 0$

$$K_f \doteqdot \overline{\bigcup_{\{\lambda \in \sigma(T)\}} \overline{B}_{Q_f}(f(\lambda))} \subseteq U, \tag{81}$$

in particular

$$\overline{B}_{Q_f}(f) \subseteq \mathcal{C}_U(\sigma(T)). \tag{82}$$

Thus $\mathcal{C}_U(\sigma(T))$ is an open set of the space $\langle \mathcal{C}(\sigma(T)), \|\cdot\|_{\sup}\rangle$, therefore we can claim that Υ is Fréchet differentiable and its differential map $\Upsilon^{[1]} : \mathcal{C}_U(\sigma(T)) \to B(\mathcal{C}(\sigma(T)))$ is such that for all $f \in \mathcal{C}_U(\sigma(T)), h \in \mathcal{C}(\sigma(T)), \lambda \in \sigma(T)$

$$\begin{cases} \Upsilon^{[1]}(f)(h)(\lambda) = \frac{d\,F}{d\lambda}(f(\lambda))h(\lambda), \\ \|\Upsilon^{[1]}(f)\|_{B(\mathcal{C}(\sigma(T)))} \leq \|\frac{d\,F}{d\lambda} \circ f\|_{\sup} \end{cases} \tag{83}$$

Let us fix $f \in \mathcal{C}_U(\sigma(T))$ and K_f as in (81), so by the boundedness of $f(\sigma(T))$ and Lemma 1.20 K_f is compact. Morever $\frac{d\,F}{d\lambda}$ is continuous on U therefore uniformly continuous on the compact K_f, hence $(\forall \varepsilon > 0)(\exists\, \delta > 0)(\forall h \in \overline{B}_{Q_f}(\mathbf{0}) \cap \overline{B}_{\delta}(\mathbf{0}))$

$$\sup_{t \in [0,1]} \sup_{\lambda \in \sigma(T)} \left| \frac{d\,F}{d\lambda}(f(\lambda) + th(\lambda)) - \frac{d\,F}{d\lambda}(f(\lambda)) \right| \leq \varepsilon, \tag{84}$$

indeed $f(\lambda) + th(\lambda) \in K_f$ and $|f(\lambda) + th(\lambda) - f(\lambda)| \leq |h(\lambda)| \leq \delta$, for all $\lambda \in \sigma(T)$ and $t \in [0,1]$. Let us identify for the moment $\mathbb{C}$ as the $\mathbb{R}-$normed space $\mathbb{R}^2$, then the usual product $(\cdot) : \mathbb{C} \times \mathbb{C} \to \mathbb{C}$ is $\mathbb{R}-$bilinear, therefore the map $F : U \subseteq \mathbb{R}^2 \to \mathbb{R}^2$ is Fréchet differentiable and for all $x \in U, h \in \mathbb{R}^2$

$$F^{[1]}(x)(h) = \frac{d\,F}{d\lambda}(x) \cdot h. \tag{85}$$

for all $h \in \overline{B}_{Q_f}(\mathbf{0})$

$$
\begin{aligned}
(86) \qquad & \sup_{\lambda\in\sigma(T)} \left|(F(f(\lambda)+h(\lambda)) - F(f(\lambda)) - \frac{dF}{d\lambda}(f(\lambda))h(\lambda)\right| = \\
& \sup_{\lambda\in\sigma(T)} \left|(F(f(\lambda)+h(\lambda)) - F(f(\lambda)) - F^{[1]}(f(\lambda))(h(\lambda)))\right| \leq \\
& \sup_{t\in[0,1]} \sup_{\lambda\in\sigma(T)} \|F^{[1]}(f(\lambda)+th(\lambda)) - F^{[1]}(f(\lambda))\|_{B(\mathbb{R}^2)} \sup_{\lambda\in\sigma(T)} |h(\lambda)| = \\
& \sup_{t\in[0,1]} \sup_{\lambda\in\sigma(T)} \left|\frac{dF}{d\lambda}(f(\lambda)+th(\lambda)) - \frac{dF}{d\lambda}(f(\lambda))\right| \|h\|_{\sup}.
\end{aligned}
$$

Here in the first equality we use (85), in the first inequality an application of the Mean value theorem applied to the Fréchet differentiable map $F : U \subset \mathbb{R}^2 \to \mathbb{R}^2$, in the second equality we use a corollary of (85). Finally by (86) and (84) $(\forall \varepsilon > 0)(\exists\, \delta > 0)(\forall h \in \overline{B}_{Q_f}(\mathbf{0}) \cap \overline{B}_\delta(\mathbf{0}) - \{\mathbf{0}\})$

$$
\frac{\sup_{\lambda\in\sigma(T)} \left|(F(f(\lambda)+h(\lambda)) - F(f(\lambda)) - \frac{dF}{d\lambda}(f(\lambda))h(\lambda)\right|}{\|h\|_{\sup}} \leq \varepsilon.
$$

Equivalently

$$
(87) \qquad \lim_{\substack{h\to\mathbf{0}\\ h\neq\mathbf{0}}} \frac{\|\Upsilon(f+h) - \Upsilon(f) - \Upsilon^{[1]}(f)(h)\|_{\sup}}{\|h\|_{\sup}} = 0
$$

Moreover

$$
\|\Upsilon^{[1]}(f)(h)\|_{\sup} \leq \|\frac{dF}{d\lambda} \circ f\|_{\sup} \|h\|_{\sup}
$$

then by (87) we proved the claimed (83). By (79), (80) and (83) we deduce that Δ is derivable in addition for all $t \in]-L, L[, \lambda \in \sigma(T)$

$$
\begin{aligned}
\frac{d\Delta}{dt}(t)(\lambda) &= \Upsilon^{[1]}(\zeta(t))(\imath)(\lambda) \\
&= \frac{dF}{d\lambda}(\zeta_t(\lambda))\imath(\lambda) = \imath \left(\frac{dF}{d\lambda}\right)_t (\lambda).
\end{aligned}
$$

Thus the claimed (78). In conclusion by the fact that $\mathbf{I}_{\mathbb{C}}^E$ is a morphism of algebras, (77), (75) and (78) for all $t \in]-L, L[$

$$
\begin{aligned}
\frac{dF(tT)}{dt} &= \frac{d}{dt}\left(\mathbf{I}_{\mathbb{C}}^E \circ \Delta\right)(t) = \mathbf{I}_{\mathbb{C}}^E\left(\frac{d\Delta}{dt}(t)\right) \\
&= \mathbf{I}_{\mathbb{C}}^E\left(\imath \cdot \left(\frac{dF}{d\lambda}\right)_t \upharpoonright \sigma(T)\right) \\
&= \mathbf{I}_{\mathbb{C}}^E(\imath)\, \mathbf{I}_{\mathbb{C}}^E\left(\left(\frac{dF}{d\lambda}\right)_t \upharpoonright \sigma(T)\right) = T\left(\frac{dF}{d\lambda}\right)_t (T).
\end{aligned}
$$

Therefore statement (2) by statement (1) applied to the map $\frac{dF}{d\lambda}$. The map $]-L, L[\ni t \mapsto \frac{dF}{d\lambda}(tT) \in B(G)$ is continuous by (72) (by replacing the map F with $\frac{dF}{d\lambda}$) hence it is Lebesgue-measurable in $B(G)$. Let $u_1, u_2 \in]-L, L[$, by statement (1) and Theorem

18.2.11. of **[DS]**

$$\begin{aligned}\int_{[u_1,u_2]}^{*}\left\|\frac{dF}{d\lambda}(tT)\right\|dt &= \int_{[u_1,u_2]}^{*}\left\|\left(\frac{dF}{d\lambda}\right)_t(T)\right\|dt\\ &\leq 4M\int_{[u_1,u_2]}^{*}\left\|\left(\frac{dF}{d\lambda}\right)_t\upharpoonright\sigma(T)\right\|_{\sup}dt\\ &\leq 4MD|u_2-u_1|<\infty,\end{aligned}$$

where $M \doteqdot \sup_{\delta\in\mathcal{B}(\mathbb{C})}\|E(\delta)\|$, and

$$D \doteqdot \sup_{t\in[u_1,u_2]}\left\|\left(\frac{dF}{d\lambda}\right)_t\upharpoonright\sigma(T)\right\|_{\sup} = \sup_{(t,\lambda)\in[u_1,u_2]\times\sigma(T)}\left|\frac{dF}{d\lambda}(t\lambda)\right|<\infty,$$

indeed $[u_1,u_2]\times\sigma(T)$ is compact and the map $(t,\lambda)\mapsto\frac{dF}{d\lambda}(t\lambda)$ is continuous on $]-L,L[\times U$. Therefore by Theorem 5, $IV.71$ of **[INT]** $]-L,L[\ni t\mapsto\frac{dF}{d\lambda}(tT)$ is Lebesgue-integrable with respect to the norm topology on $B(G)$, so in particular by Definition 1, $IV.33$ of **[INT]**

$$\exists\oint_{u_1}^{u_2}\frac{dF}{d\lambda}(tT)dt\in B(G). \tag{88}$$

Therefore by (8), (88), Theorem 1, $IV.35$ of **[INT]** and (72)

$$T\oint_{u_1}^{u_2}\frac{dF}{d\lambda}(tT)dt=\oint_{u_1}^{u_2}T\frac{dF}{d\lambda}(tT)dt=\oint_{u_1}^{u_2}\frac{dF(tT)}{dt}dt. \tag{89}$$

By (72) the map $]-L,L[\ni t\mapsto F(tT)$, is derivable moreover its derivative $]-L,L[\ni t\mapsto\frac{dF(tT)}{dt}$ is continuous in $B(G)$ by (72) and the continuity of the map $]-L,L[\ni t\mapsto\frac{dF}{d\lambda}(tT)$ in $B(G)$. Therefore $[u_1,u_2]\ni t\mapsto\frac{dF(tT)}{dt}$ is Lebesgue integrable in $B(G)$, where the integral has to be understood as defined in Ch II of **[FVR]**, see Proposition 3, $n^{\circ}3$, §1, Ch II of **[FVR]**. Finally the Lebesgue integral for functions with values in a Banach space as defined in Ch II of **[FVR]**, turns to be the integral with respect to the Lebesgue measure as defined in Ch. IV, §4, $n^{\circ}1$ of **[INT]** (see Ch III, §3, $n^{\circ}3$ and example in Ch IV, §4, $n^{\circ}4$ of **[INT]**). Thus statement (3) follows by (89). □

LEMMA 1.22. *Let R be a possibly unbounded scalar type spectral operator in G, $\sigma(R)$ its spectrum, E its resolution of the identity, $K\neq\emptyset$ and for all $t\in K$ be $f_t\in Bor(\sigma(R))$ such that*

$$N\doteqdot\sup_{t\in K}\|\widetilde{f}_t\|_{\infty}^{E}<\infty. \tag{90}$$

If $g\in Bor(\sigma(R))$ and $\{\sigma_n\}_{n\in\mathbb{N}}$ is an $E-$sequence then for all $v\in Dom(g(R))$

$$\lim_{n\in\mathbb{N}}\sup_{t\in K}\|f_t(R)g(R)v-f_t(R)g(R)E(\sigma_n)v\|=0.$$

PROOF. By statement (g) of Theorem $18.2.11.$ of **[DS]** the statement is well set. Let $M\doteqdot\sup_{\sigma\in\mathcal{B}(\mathbb{C})}\|E(\sigma)\|_{B(G)}$ then $M<\infty$ by Corollary $15.2.4.$ of **[DS]**. Hypothesis (90) together statement (c) of Theorem $18.2.11.$ of **[DS]**, imply that for all $t\in K$, $f_t(R)\in B(G)$ and

$$\sup_{t\in K}\|f_t(R)\|_{B(G)}\leq 4MN.$$

Therefore for all $v \in Dom(g(R))$ we have

$$\begin{aligned}
&\lim_{n\in\mathbb{N}} \sup_{t\in K} \|f_t(R)g(R)v - f_t(R)g(R)E(\sigma_n)v\| \\
&\leq \lim_{n\in\mathbb{N}} \sup_{t\in K} \|f_t(R)\| \cdot \|g(R)v - g(R)E(\sigma_n)v\| \\
&\leq 4MN \lim_{n\in\mathbb{N}} \|g(R)v - g(R)E(\sigma_n)v\| \\
&= 4MN \lim_{n\in\mathbb{N}} \|g(R)v - E(\sigma_n)g(R)v\| && \text{by } (g) \text{ of Theorem } 18.2.11. \text{ of } [\mathbf{DS}] \\
&= 0 && \text{by (47).}
\end{aligned}$$

□

THEOREM 1.23 (**Strong operator derivability**). *Let R be a possibly unbounded scalar type spectral operator in G, $K \subseteq \mathbb{R}$ an open interval of $\mathbb{R}$ and U an open neighbourhood of $\sigma(R)$ such that $K \cdot U \subseteq U$. Assume that $f : U \to \mathbb{C}$ is an analytic map and*

$$\sup_{t\in K} \|\widetilde{f}_t\|_\infty^E < \infty.$$

Then

(1) *the map $K \ni t \mapsto f(tR) \in B(G)$ is continuous in the strong operator topology,*
(2) *if*

$$\sup_{t\in K} \left\| \widetilde{\left(\frac{d\,f}{d\,\lambda}\right)}_t \right\|_\infty^E < \infty, \tag{91}$$

then for all $v \in Dom(R), t \in K$

$$\frac{df(tR)v}{dt} = R\frac{df}{d\lambda}(tR)v \in G.$$

PROOF. Let $\{\sigma_n\}_{n\in\mathbb{N}}$ be an $E-$sequence of compact sets, then by Lemma 1.22 applied for the Borelian map $g : \sigma(R) \ni \lambda \to 1 \in \mathbb{C}$, so $g(R) = \mathbf{1}$, and by (71) we have for all $v \in G$

$$\lim_{n\in\mathbb{N}} \sup_{t\in K} \|f(tR)v - f(tR)E(\sigma_n)v\| = 0. \tag{92}$$

By (71) and Key Lemma 1.7 for all $n \in \mathbb{N}$

$$\begin{aligned} f(tR)E(\sigma_n) &= f_t(R)E(\sigma_n) = f_t(R_{\sigma_n} \upharpoonright G_{\sigma_n})E(\sigma_n) \\ &= f(t(R_{\sigma_n} \upharpoonright G_{\sigma_n}))E(\sigma_n). \end{aligned} \tag{93}$$

σ_n is bounded so by Key Lemma 1.7 $R_{\sigma_n} \upharpoonright G_{\sigma_n}$ is a scalar type spectral operator such that $R_{\sigma_n} \upharpoonright G_{\sigma_n} \in B(G_{\sigma_n})$, moreover by (43) U is an open neighbourhood of $\sigma(R_{\sigma_n} \upharpoonright G_{\sigma_n})$. Thus by statement (2) of Theorem 1.21 the map

$$K \ni t \mapsto f(t(R_{\sigma_n} \upharpoonright G_{\sigma_n})) \in B(G_{\sigma_n})$$

is derivable, so in particular $\|\cdot\|_{B(G_{\sigma_n})}-$continuous. Now for all $n \in \mathbb{N}, v_n \in G_{\sigma_n}$ define $\xi_{v_n} : B(G_{\sigma_n}) \ni A \mapsto Av_n \in G$, then $\xi_{v_n} \in B(B(G_{\sigma_n}), G)$, thus as a composition of two continuous maps also the following map

$$K \ni t \mapsto \xi_{E(\sigma_n)v}\left(f(t(R_{\sigma_n} \upharpoonright G_{\sigma_n}))\right) \in G \tag{94}$$

is $\|\cdot\|_G-$continuous, for all $n \in \mathbb{N}, v \in G$. Hence by (93) we have for all $n \in \mathbb{N}$

$$K \ni t \mapsto f(tR)E(\sigma_n) \in B(G) \text{ is strongly continuous.} \tag{95}$$

Finally by (95) and (92) we can apply Theorem 2, $\S 1.6.$, Ch. 10 of [**GT**] to the uniform space $B(G)_{st}$ whose uniformity is generated by the set of seminorms defining the strong operator topology on $B(G)$. Thus we conclude that $K \ni t \mapsto f(tR) \in B(G)$ is strongly continuous, and statement (1) follows. Let $n \in \mathbb{N}$ and $v_n \in G_{\sigma_n}$ so $\xi_{v_n} \in B(B(G_{\sigma_n}), G)$ thus ξ_{v_n} is Fréchet differentiable with constant differential map $\xi_{v_n}^{[1]} : B(G_{\sigma_n}) \ni A \mapsto \xi_{v_n} \in B(B(G_{\sigma_n}), G)$. Therefore by statement (2) of Theorem 1.21 for all $n \in \mathbb{N}, v \in G$ the map in (94) is Fréchet differentiable as composition of two Fréchet differentiable maps, and its derivative is for all $t \in K$

$$
\begin{aligned}
\frac{d}{dt}\left(f(t(R_{\sigma_n} \restriction G_{\sigma_n}))E(\sigma_n)v\right) &= \xi_{E(\sigma_n)v}\left(\frac{d}{dt}(f(t(R_{\sigma_n} \restriction G_{\sigma_n})))\right) \\
&= \frac{d}{dt}(f(t(R_{\sigma_n} \restriction G_{\sigma_n})))E(\sigma_n)v \\
&= (R_{\sigma_n} \restriction G_{\sigma_n})\frac{df}{d\lambda}(t(R_{\sigma_n} \restriction G_{\sigma_n}))E(\sigma_n)v, && \text{by (72)} \\
&= \frac{df}{d\lambda}(t(R_{\sigma_n} \restriction G_{\sigma_n}))(R_{\sigma_n} \restriction G_{\sigma_n})E(\sigma_n)v, && \text{by 18.2.11., [DS]} \\
&= \left(\frac{df}{d\lambda}\right)_t (R_{\sigma_n} \restriction G_{\sigma_n})(R_{\sigma_n} \restriction G_{\sigma_n})E(\sigma_n)v && \text{by (71)} \\
&= \left(\frac{df}{d\lambda}\right)_t (R)(R_{\sigma_n} \restriction G_{\sigma_n})E(\sigma_n)v && \text{by Lemma 1.7} \\
&= \frac{df}{d\lambda}(tR)(R_{\sigma_n} \restriction G_{\sigma_n})E(\sigma_n)v. && \text{by (71)}
\end{aligned} \tag{96}
$$

Thus by (93) for all $n \in \mathbb{N}, v \in G$

$$
\begin{cases} K \ni t \mapsto f(tR)E(\sigma_n)v \in G \text{ is differentiable and} \\ K \ni t \mapsto \frac{df}{d\lambda}(tR)(R_{\sigma_n} \restriction G_{\sigma_n})E(\sigma_n)v \in G \text{ is its derivative.} \end{cases} \tag{97}
$$

By (91) we can apply Lemma 1.22 to the maps $\left(\frac{df}{d\lambda}\right)_t \restriction \sigma(R)$ and $g = \imath : \sigma(R) \ni \lambda \mapsto \lambda \in \mathbb{C}$, so $g(R) = R$, hence by (71) for all $v \in Dom(R)$

$$
\lim_{n\in\mathbb{N}} \sup_{t\in K} \left\| \frac{df}{d\lambda}(tR)Rv - \frac{df}{d\lambda}(tR)(R_{\sigma_n} \restriction G_{\sigma_n})E(\sigma_n)v \right\| = 0. \tag{98}
$$

Moreover for all $a \in K$ let $r_a \in \mathbb{R}^+$ be such that $B_{r_a}(a) \subset K$ which exists K being open, then the equations (98), (97) and (92) hold again if we replace K by $B_{r_a}(a)$. Hence we can apply Theorem 8.6.3. of [**Dieu**] and deduce for all $v \in Dom(R)$ that the map $K \ni t \mapsto f(tR)v \in G$ is derivable, and its derivative map is

$$
K \ni t \mapsto \frac{df}{d\lambda}(tR)Rv \in G.
$$

Finally for all $v \in Dom(R)$, $R\frac{df}{d\lambda}(tR)v = \frac{df}{d\lambda}(tR)Rv$, by $Dom(\frac{df}{d\lambda}(tR)) = G$ and the commutativity property of the Borel functional calculus expressed in statement (f) of Theorem 18.2.11. of [**DS**]. Hence the statement follows. □

COROLLARY 1.24. *Let R be a possibly unbounded scalar type spectral operator in G, U an open neighbourhood of $\sigma(R)$ and $S : U \to \mathbb{C}$ an analytic map. Assume that there is $L > 0$ such that $]-L, L[\cdot U \subseteq U$ and*

(1) $\widetilde{S}_t \in \mathfrak{L}_E^\infty(\sigma(R))$ *and* $\widetilde{\left(\frac{dS}{d\lambda}\right)_t} \in \mathfrak{L}_E^\infty(\sigma(R))$ *for all* $t \in]-L, L[$;

(2)
$$\int^{*}\left\|\widetilde{\left(\frac{d\,S}{d\,\lambda}\right)_t}\right\|_{\infty}^{E} dt<\infty$$
(here the upper integral is with respect to the Lebesgue measure on $]-L,L[$*);*

(3) *for all* $v\in G$ *the map* $]-L,L[\ni t\mapsto \frac{d\,S}{d\,\lambda}(tR)v\in G$ *is Lebesgue measurable.*

Then for all $u_1,u_2\in]-L,L[$
$$R\int_{u_1}^{u_2}\frac{d\,S}{d\,\lambda}(tR)\,d\,t=S(u_2R)-S(u_1R)\in B(G).$$
Here the integral is with respect to the Lebesgue measure on $[u_1,u_2]$ *and with respect to the strong operator topology on* $B(G)$*, see Definition 1.12.*

PROOF. Let $M\doteqdot\sup_{\sigma\in\mathcal{B}(\mathbb{C})}\|E(\sigma)\|_G$ and μ the Lebesgue measure on $[u_1,u_2]$, then by (71), hypotheses, and statement (c) of Theorem 18.2.11 of [**DS**] we have

a: for all $t\in[u_1,u_2]$, $S(tR)\in B(G)$;

b: for all $t\in[u_1,u_2]$, $\frac{d\,S}{d\,\lambda}(tR)\in B(G)$;

c: $([u_1,u_2]\ni t\mapsto\|\frac{d\,S}{d\,\lambda}(tR)\|_{B(G)})\in\mathfrak{F}_1([u_1,u_2];\mu)$.

So by hypothesis (3), the (c) and Remark 1.14 we have that the map
$$[u_1,u_2]\ni t\mapsto\frac{d\,S}{d\,\lambda}(tR)\in B(G)$$
is Lebesgue integrable with respect to the strong operator topology. This means that, except for (59), the hypotheses of Theorem 1.18 hold for $X\doteqdot[u_1,u_2]$, $h\doteqdot(S_{u_2}-S_{u_1})\upharpoonright\sigma(R)$, $g:\sigma(R)\ni\lambda\mapsto\lambda\in\mathbb{C}$ and finally for the maps $f_t\doteqdot\left(\frac{d\,S}{d\,\lambda}\right)_t\upharpoonright\sigma(R)$, for all $t\in[u_1,u_2]$. Next let $\sigma\in\mathcal{B}(\mathbb{C})$ be bounded, so by Key Lemma 1.7 $R_\sigma\upharpoonright G_\sigma$ is a scalar type spectral operator such that $R_\sigma\upharpoonright G_\sigma\in B(G_\sigma)$, moreover by (43) U is an open neighbourhood of $\sigma(R_\sigma\upharpoonright G_\sigma)$. Thus we can apply statement (3) of Theorem 1.21 to the Banach space G_σ, the analytic map S and to the operator $R_\sigma\upharpoonright G_\sigma$. In particular the map $[u_1,u_2]\ni t\mapsto\frac{d\,S}{d\,\lambda}(t(R_\sigma\upharpoonright G_\sigma))\in B(G_\sigma)$ is Lebesgue integrable in $\|\cdot\|_{B(G_\sigma)}-$topology, that is in the meaning of Definition 2, $IV.23$ of [**INT**]. Next we consider for all $v\in G_\sigma$, the following map
$$T\in B(G_\sigma)\mapsto Tv\in G_\sigma$$
which is linear and continuous in the norm topologies. Thus by Theorem 1, $IV.35$ of the [**INT**], $[u_1,u_2]\ni t\mapsto\frac{d\,S}{d\,\lambda}(t(R_\sigma\upharpoonright G_\sigma))v\in G_\sigma$ is Lebesgue integrable for all $v\in G_\sigma$ and
$$\int_{u_1}^{u_2}\frac{d\,S}{d\,\lambda}(t(R_\sigma\upharpoonright G_\sigma))v\,d\,t=\left(\oint_{u_1}^{u_2}\frac{d\,S}{d\,\lambda}(t(R_\sigma\upharpoonright G_\sigma))\,d\,t\right)v.$$
Therefore by Definition 1.12 we can state that $[u_1,u_2]\ni t\mapsto\frac{d\,S}{d\,\lambda}(t(R_\sigma\upharpoonright G_\sigma))\in B(G_\sigma)$ is Lebesgue integrable with respect to the strong operator topology on $B(G_\sigma)$ and
$$\int_{u_1}^{u_2}\frac{d\,S}{d\,\lambda}(t(R_\sigma\upharpoonright G_\sigma))\,d\,t=\oint_{u_1}^{u_2}\frac{d\,S}{d\,\lambda}(t(R_\sigma\upharpoonright G_\sigma))\,d\,t.\tag{99}$$
Here $\int_{u_1}^{u_2}\frac{d\,S}{d\,\lambda}(t(R_\sigma\upharpoonright G_\sigma))dt$ is the integral of $\frac{d\,S}{d\,\lambda}(t(R_\sigma\upharpoonright G_\sigma))$ with respect to the Lebesgue measure on $[u_1,u_2]$ and the strong operator topology on $B(G_\sigma)$. Furthermore by statement (3) of Theorem 1.21
$$(R_\sigma\upharpoonright G_\sigma)\oint_{u_1}^{u_2}\frac{d\,S}{d\,\lambda}(t(R_\sigma\upharpoonright G_\sigma))\,d\,t=S(u_2(R_\sigma\upharpoonright G_\sigma))-S(u_1(R_\sigma\upharpoonright G_\sigma)).$$

Thus by (99)

$$(R_\sigma \upharpoonright G_\sigma)\int_{u_1}^{u_2} \frac{d\,S}{d\,\lambda}(t(R_\sigma \upharpoonright G_\sigma))\,d\,t = S(u_2(R_\sigma \upharpoonright G_\sigma)) - S(u_1(R_\sigma \upharpoonright G_\sigma)). \tag{100}$$

Which implies (59), by choosing for example $\sigma_n \doteqdot B_n(\mathbf{0})$, for all $n \in \mathbb{N}$. Therefore by Theorem 1.18 we obtain the statement. $\square$

THEOREM 1.25 (**Strong operator Newton-Leibnitz formula**). *Let R be a possibly unbounded scalar type spectral operator in G, U an open neighbourhood of $\sigma(R)$ and $S : U \to \mathbb{C}$ an analytic map. Assume that there is $L > 0$ such that $]-L, L[\cdot U \subseteq U$ and*

(1) $\widetilde{S}_t \in \mathfrak{L}_E^\infty(\sigma(R))$ *for all* $t \in]-L, L[$;

(2)

$$\sup_{t \in]-L,L[} \left\| \left(\widetilde{\frac{d\,S}{d\,\lambda}}\right)_t \right\|_\infty^E < \infty.$$

Then

(1) *for all* $u_1, u_2 \in]-L, L[$

$$R\int_{u_1}^{u_2} \frac{d\,S}{d\,\lambda}(tR)\,d\,t = S(u_2 R) - S(u_1 R) \in B(G).$$

Here the integral is with respect to the Lebesgue measure on $[u_1, u_2]$ and with respect to the strong operator topology on $B(G)$.

(2) *If also* $\sup_{t\in]-L,L[} \left\|\widetilde{S}_t\right\|_\infty^E < \infty$, *then for all* $v \in Dom(R), t \in]-L, L[$

$$\frac{dS(tR)v}{dt} = R\frac{dS}{d\lambda}(tR)v.$$

PROOF. By hypothesis (2) and statement (1) of Theorem 1.23 for all $v \in G$ the map $]-L, L[\ni t \mapsto \frac{d\,S}{d\,\lambda}(tR)v \in G$ is continuous. Thus statement (1) by Corollary 1.24 and the fact that continuity implies measurability. Statement (2) follows by statement (2) of Theorem 1.23. $\square$

REMARK 1.26. We end this section by remarking that $f : X \to B(G)$ is $\mu-$integrable with respect to the strong operator topology as defined in Definition 1.12, if and only if $f : X \to B(G)$ is scalarly $(\mu, B(G))-$integrable with respect to the weak operator topology in the sense explained in Notations 2.1. In Chapter 2 we shall extend the results of Chapter 1 to the case of integration with respect to the measure μ and with respect to the $\sigma(B(G), \mathcal{N})-$topology, where $\mathcal{N} \subset B(G)^*$ is a suitable linear subspace of the topological dual of $B(G)$.

4. Application to resolvents of unbounded scalar type spectral operators in a Banach space G

COROLLARY 1.27. *Let T be a possibly unbounded scalar type spectral operator in G with real spectrum $\sigma(T)$. Then*

(1) *for all* $\lambda \in \mathbb{C} \mid Im(\lambda) > 0$

$$(T - \lambda\mathbf{1})^{-1} = i\int_{-\infty}^{0} e^{-it\lambda}e^{itT}\,d\,t \in B(G). \tag{101}$$

(2) *for all* $v \in Dom(T), t \in \mathbb{R}$
$$\frac{d\, e^{it(T-\lambda\mathbf{1})}v}{d\,t} = i(T-\lambda\mathbf{1})e^{i(T-\lambda\mathbf{1})t}v.$$

REMARK 1.28. If we set the map $S(\lambda) \doteqdot \exp(i\lambda)$ for all $\lambda \in \mathbb{C}$ then the operator functions in Corollary 1.27 are so defined $e^{itT} \doteqdot S_t(T)$ and $e^{it(T-\lambda\mathbf{1})} \doteqdot S_t(T-\lambda\mathbf{1})$, in the sense of the Borelian functional calculus for the scalar type spectral operators T and $(T-\lambda\mathbf{1})$, respectively, as defined in Definition 1.3.

The integral in Corollary 1.27 is with respect to the Lebesgue measure and with respect to the strong operator topology on $B(G)$. Meaning by definition that
$$\int_{-\infty}^{0} e^{-it\lambda}e^{itT}\,d\,t \in B(G)$$
such that for all $v \in G$
$$\left(\int_{-\infty}^{0} e^{-it\lambda}e^{itT}\,d\,t\right)v \doteqdot \lim_{u\to-\infty}\left(\int_{u}^{0} e^{-it\lambda}e^{itT}\,d\,t\right)v = \lim_{u\to-\infty}\int_{u}^{0} e^{-it\lambda}e^{itT}v\,d\,t.$$
Here the integral in the right side of the first equality is with respect to the Lebesgue measure on $[u,0]$ and with respect to the strong operator topology on $B(G)$.

PROOF. Let $\lambda \in \mathbb{C}$ and set $R \doteqdot (T-\mathbf{1}\lambda)$, then R is a scalar type spectral operator, see Theorem 18.2.17. of the [**DS**]. Let $\lambda \in \mathbb{C} \mid Im(\lambda) \neq 0$ and E be the resolution of the identity of R, then $\sigma(R) = \sigma(T) - \lambda$, as a corollary of the well-known spectral mapping theorem. Then for all $t \in \mathbb{R}$
$$\begin{aligned} E-ess\sup_{\nu\in\sigma(R)}\left|\frac{dS}{d\lambda}(t\nu)\right| &= E-ess\sup_{\nu\in\sigma(R)}|S(t\nu)| \leq \\ &\leq \sup_{\nu\in\sigma(R)}|S(t\nu)| \\ &= \sup_{\mu\in\sigma(T)}|e^{i(\mu-\lambda)t}| \\ &= e^{Im(\lambda)t}. \end{aligned}$$
Therefore are verified the hypotheses of Corollary 1.25 with the position $R \doteqdot (T-\lambda\mathbf{1})$, then we can state for all $v \in G, u \in \mathbb{R}$ that
$$i(T-\lambda\mathbf{1})\int_{u}^{0} e^{it(T-\lambda\mathbf{1})}v\,d\,t = v - e^{iu(T-\lambda\mathbf{1})}v. \tag{102}$$
Here $e^{it(T-\lambda\mathbf{1})} \doteqdot S_t(R)$. One should note an apparent ambiguity about the symbol $e^{it(T-\lambda\mathbf{1})}$, standing here for the operator $S_t(R) = S(tR)$, which could be seen also as a Borelian function of the operator T. By setting $g^{[\lambda]}(\mu) \doteqdot \mu-\lambda$, so $g^{[\lambda]} = \imath - \lambda\cdot\mathbf{1}$ with $\mathbf{1} : \mathbb{C} \ni \lambda \mapsto 1$, considering that by the composition rule, see Theorem $18.2.24$ of [**DS**], we have $S_t \circ g^{[\lambda]}(T) = S_t(g^{[\lambda]}(T))$, finally $R = \imath(T) - \lambda\mathbf{1}(T) = (\imath - \lambda\cdot\mathbf{1})(T) = g^{[\lambda]}(T)$, we can assert
$$\begin{cases} T-\lambda\mathbf{1} = g^{[\lambda]}(T) \doteqdot T-\lambda \\ e^{it(T-\lambda\mathbf{1})} \doteqdot S_t(T-\lambda\mathbf{1}) = S_t\circ g^{[\lambda]}(T) = e^{it(T-\lambda)}. \end{cases} \tag{103}$$
Therefore we can consider the operator $e^{it(T-\lambda\mathbf{1})}$ as an operator function of R or of T. Now for all $t \in \mathbb{R}$, $\sup_{\mu\in\sigma(T)}|\exp(i\mu t)| = 1$, therefore we can deduce by statement (c)

of Theorem 18.2.11. of [**DS**]

$$\sup_{t\in\mathbb{R}} \| \exp(iTt)\|_{B(G)} \leq 4M. \tag{104}$$

Here $M \doteqdot \sup_{\sigma\in\mathcal{B}(\mathbb{C})} \|E(\sigma)\|_G$. But with the notations before adopted we have for all $\mu \in \mathbb{C}$ that $S_t \circ g^{[\lambda]}(\mu) = \exp(it(\mu-\lambda)) = \exp(-it\lambda)S_t(\mu)$, therefore by considering that $S_t(T) = S(tT)$, see (71), we have $S_t \circ g^{[\lambda]}(T) = \exp(-it\lambda)S_t(T) = \exp(-it\lambda)S(tT)$. Thus by (103) we have for all $t \in \mathbb{R}, \lambda \in \mathbb{C} \mid Im(\lambda) > 0$

$$e^{it(T-\lambda\mathbf{1})} = \exp(-it\lambda)S(tT) \doteq \exp(-it\lambda)e^{itT}. \tag{105}$$

We have by (105) and (104)

$$\lim_{u\to-\infty} \|e^{iu(T-\lambda\mathbf{1})}\|_{B(G)} \leq 4M \lim_{u\to-\infty} \exp(Im(\lambda)u) = 0$$

or equivalently $\lim_{u\to-\infty} e^{iu(T-\lambda\mathbf{1})} = \mathbf{0}$ in $\|\cdot\|_{B(G)}-$topology. Hence by (102) for all $v \in G$

$$v = i \lim_{u\to-\infty} (T-\lambda\mathbf{1}) \int_u^0 e^{it(T-\lambda\mathbf{1})} v \, dt \text{ in } \|\cdot\|_G. \tag{106}$$

By considering that $Im(\lambda) \neq 0$ we have $\{\mu \in \mathbb{C} \mid g^{[\lambda]}(\mu) = 0\} \cap \sigma(T) = \emptyset$, therefore if we denote with F the resolution of the identity of the spectral operator T, we have $F(\sigma(T)) = \mathbf{1}$ so $F(\{\mu \in \mathbb{C} \mid g^{[\lambda]}(\mu) = 0\}) = F(\{\mu \in \mathbb{C} \mid g^{[\lambda]}(\mu) = 0\} \cap \sigma(T)) = F(\emptyset) \doteqdot \mathbf{0}$. Thus by applying statement (h) of Theorem 18.2.11. of [**DS**], we can assert that

$$\exists\, (T-\lambda)^{-1} = \frac{1}{g^{[\lambda]}}(T) \doteq \frac{1}{T-\lambda}.$$

Finally $F - ess\sup_{\mu\in\sigma(T)} |\frac{1}{g^{[\lambda]}(\mu)}| \leq \sup_{\mu\in\sigma(T)} |\frac{1}{g^{[\lambda]}(\mu)}| = \sup_{\mu\in\sigma(T)} |\frac{1}{\mu-\lambda}| = \frac{1}{\inf_{\mu\in\sigma(T)} |(\mu-\lambda)|} \leq \frac{1}{|Im(\lambda)|} < \infty$, so

$$\frac{1}{g^{[\lambda]}}(T) \in B(G).$$

Hence by the previous equation and the fact $T - \lambda = T - \lambda\mathbf{1}$, see (103), we can state

$$(T-\lambda\mathbf{1})^{-1} \in B(G).$$

Finally by following a standard argument, see for example [**LN**], by this one and (106) we can deduce for all $v \in G$ that

$$\begin{aligned}(T-\lambda\mathbf{1})^{-1}v &= i \lim_{u\to-\infty} (T-\lambda\mathbf{1})^{-1}(T-\lambda\mathbf{1}) \int_u^0 e^{it(T-\lambda\mathbf{1})} v \, dt \\ &= i \lim_{u\to-\infty} \int_u^0 e^{it(T-\lambda\mathbf{1})} v \, dt.\end{aligned}$$

So statement (1) by (105). By (105), the fact that $S_t(T) = S(tT)$ and statement (2) of Theorem 1.23 applied to the operator T and to the map $S : \mathbb{C} \ni \mu \mapsto e^{i\mu}$, we obtain statement (2). □

REMARK 1.29. An important application of this formula is made in proving the well-known Stone theorem for strongly continuous semigroups of unitary operators in Hilbert space, see Theorem 12.6.1. of [**DS**]. In [**Dav**] it has been used for showing the equivalence of uniform convergence in strong operator topology of a one-parameter semigroup depending on a parameter and the convergence in strong operator topology of the resolvents of the corresponding generators, Theorem 3.17..

Notice that if $\zeta \doteqdot -i\lambda$ and $Q \doteqdot iT$, then the equality (101) turns into

$$(Q + \zeta \mathbf{1})^{-1} = \int_0^\infty e^{-t\zeta} e^{-Qt}\, d\, t,$$

which is referred in $IX.1.3.$ of [**Kat**] as the fact that the resolvent of Q is the *Laplace* transform of the semigroup e^{-Qt}. Applications of this formula to perturbation theory are in $IX.2.$ of [**Kat**].

CHAPTER 2

Extension theorem. The case of the topology $\sigma(B(G), \mathcal{N})$

1. Introduction

Let R be an unbounded scalar type spectral operator R in a complex Banach space G and E its resolution of identity. The main results of this chapter and of the work are of two types.

The results of the first type are Extension Theorems for integration with respect to the $\sigma(B(G), \mathcal{N})-$topology, when $\mathcal{N}$ is an $E-$appropriate set: Theorems 2.25 and when $\mathcal{N}$ is an $E-$appropriate set with the duality property: Corollary 2.26.

As an application we will prove, by using (143), the Extension theorems for the integration with respect to the sigma-weak topology: Corollary 2.28 and Corollary 2.29, and for integration with respect to the weak operator topology: Corollary 2.27, and Corollary 2.30.

The results of the second type are Newton-Leibnitz formulas for integration with respect to the $\sigma(B(G), \mathcal{N})-$topology, when $\mathcal{N}$ is an $E-$appropriate set with the duality property: Corollary 2.33 and Corollary 2.34; for integration with respect to the sigma-weak topology: Corollary 2.35; for integration with respect to the weak operator topology: Corollary 2.36

For obtaining the Extension Theorem 2.25 we need to introduce the concept of $E-$appropriate set, Definition 2.11, which allows us to establish two important properties for the proof of Theorem 2.25, namely the “Commutation” property, Theorem 2.13, and the “Restriction” property, Theorem 2.22.

Finally for obtaining Corollary 2.26 and the Newton-Leibnitz formula in Corollary 2.33 we have to introduce the concept of an $E-$appropriate set $\mathcal{N}$ with the duality property, Definition 2.11, which allows us to establish conditions ensuring that a map is scalarly essentially $(\mu, B(G))-$integrable with respect to the $\sigma(B(G), \mathcal{N})-$topology, Theorem 2.2. Similar results for the weak operator topology are contained in Theorem 2.5 and Corollary 2.6.

2. Existence of the weak-integral with respect to the $\sigma(B(G), \mathcal{N})-$ topology

In this section we shall obtain a general result, Theorem 2.2 about conditions ensuring that a map is scalarly essentially $(\mu, B(G))-$integrable with respect to the $\sigma(B(G), \mathcal{N})-$topology, where $\mathcal{N}$ is a suitable subset of $B(G)^*$.

NOTATIONS WITH COMMENTS 2.1. Let $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$, Z a linear space over $\mathbb{K}$ and τ a locally convex topology on Z, then we indicate with $\langle Z, \tau \rangle$ the associated locally convex space over $\mathbb{K}$. We denote with $LCS(\mathbb{K})$ the class of all the locally convex spaces over $\mathbb{K}$ and for any $\langle Z, \tau \rangle \in LCS(\mathbb{K})$ we set $\langle Z, \tau \rangle^*$ for its topological dual, that is the $\mathbb{K}-$linear space of all $\mathbb{K}-$linear continuous functionals on Z.

Let Y be a linear space over $\mathbb{K}$ and U a subspace of $Hom(Y, \mathbb{K})$, then we indicate with the symbol $\sigma(Y, U)$ the weakest (locally convex) topology on Y such that

$U \subseteq \langle Y, \sigma(Y, U)\rangle^*$, Def. 2, $II.42$ of [**TVS**], which coincides with the locally convex topology on Y generated by the set of seminorms $\Gamma(U)$ associated to U where $\Gamma(U) \doteqdot \{q_\phi : Y \ni y \mapsto |\phi(y)| \mid \phi \in U\}$.

It is not hard to see that $\sigma(Y, U)$ is the topology generated by the set of seminorms $\Gamma(S)$ for any S such that $U = \mathfrak{L}_{\mathbb{K}}(S)$, where $\mathfrak{L}_{\mathbb{K}}(S)$ is the $\mathbb{K}-$linear space generated by the set S.

By Proposition 2, $II.43$ of [**TVS**], $\sigma(Y, U)$ is a Hausdorff topology if and only if U separates the points of Y, i.e.

$$(\forall T \in Y)(T \neq \mathbf{0} \Rightarrow (\exists\, \phi \in U)(\phi(T) \neq 0)). \tag{107}$$

Also by Proposition 3, $II.43$ of [**TVS**]

$$\langle Y, \sigma(Y, U)\rangle^* = U. \tag{108}$$

Let X be a locally compact space and μ a $\mathbb{K}-$ Radon measure on X, Definition 2, §1, $n^\circ 3$, Ch. 3, of [**INT**] where it is called just measure. We denote with $|\mu|$ the total variation of μ, §1, $n^\circ 6$, Ch. 3, of [**INT**], and with $\int^*$ the upper integral with respect to a positive measure, as for example $|\mu|$, Definition 1, §1, $n^\circ 1$, Ch. 4, of [**INT**], With $\int^\bullet$ we denote the essential upper integral with respect to a positive measure, Definition 1, §1, $n^\circ 1$, Ch. 5, of [**INT**]. [1] We readdress for the definition of essentially $\mu-$integrable map $f : X \to \mathbb{K}$, to Ch. 5, §1, $n^\circ 3$, of [**INT**].

Let $\langle Y, \tau\rangle \in LCS(\mathbb{K})$ of Hausdorff then $f : X \to \langle Y, \tau\rangle$ is *scalarly essentially $\mu-$integrable* or equivalently $f : X \to Y$ is *scalarly essentially $\mu-$integrable with respect to the measure μ and with respect to the $\tau-$ topology on Y* if for all $\omega \in \langle Y, \tau\rangle^*$ the map $\omega \circ f : X \to \mathbb{K}$ is essentially $\mu-$integrable, so we can define its *integral* as the following linear operator

$$\langle Y, \tau\rangle^* \ni \omega \mapsto \int \omega(f(x))\, d\mu(x) \in \mathbb{K}.$$

See Ch. 6, §1, $n^\circ 1$ for $\mathbb{K} = \mathbb{R}$, and for the extension to the case $\mathbb{K} = \mathbb{C}$ see the end of §2, $n^\circ 10$, of [**INT**].

Notice that the previous definitions depend only on the dual space $\langle Y, \tau\rangle^*$, hence both the concepts of scalar essential $\mu-$integrability and integral will be invariant if we replace τ with any other Hausdorff locally convex topology τ_2 on Y compatible with the duality $(Y, \langle Y, \tau\rangle^*)$, i.e. such that $\langle Y, \tau\rangle^* = \langle Y, \tau_2\rangle^*$.

Therefore as a corollary of the well-known Mackey-Arens Theorem, see Theorem 1, $IV.2$ of [**TVS**] or Theorem 5 §8.5. of [**Jar**], fixed a locally convex space $\langle Y, \tau\rangle$ and denoted by $\mathcal{N} \doteqdot \langle Y, \tau\rangle^*$ its topological dual, we have that scalar essential $\mu-$integrability (respectively integral) is an invariant property (respectively functional) under the variation of any Hausdorff locally convex topology τ_1 on Y such that

$$\sigma(Y, \mathcal{N}) \leq \tau_1 \leq \tau(Y, \mathcal{N}).$$

Here $a \leq b$ means a is weaker than b and $\tau(Y, \mathcal{N})$ is the Mackey topology associated to the canonical duality $(Y, \mathcal{N})$.

Let $f : X \to \langle Y, \tau\rangle$ be scalarly essentially $\mu-$integrable and assume that

$$(\exists\, B \in Y)(\forall \omega \in \langle Y, \tau\rangle^*)\left(\omega(B) = \int \omega(f(x))\, d\mu(x)\right). \tag{109}$$

[1] In general $\int^\bullet \leq \int^*$, however if X is $\sigma-$compact, in particular compact, then $\int^\bullet = \int^*$.

Notice that by the Hahn-Banach theorem $\langle Y,\tau\rangle^*$ separates the points of Y, so the element B is defined by this condition uniquely. In this case, by definition $f : X \to \langle Y,\tau\rangle$ **is scalarly essentially** $(\mu, Y)-$**integrable** (or $f : X \to Y$ **is scalarly essentially** $(\mu, Y)-$**integrable with respect to the** $\tau-$ **topology on** Y) and its **weak-integral with respect to the measure** μ **and with respect to the** $\tau-$**topology**, or briefly its **weak-integral**, is defined by

$$\int f(x)\,d\mu(x) \doteqdot B. \tag{110}$$

In the work we shall use this integral for the case $\langle Y,\tau\rangle \doteqdot \langle B(G),\sigma(B(G),\mathcal{N})\rangle$, where $\mathcal{N}$ is a linear subspace of $B(G)^*$ which separates the points of $B(G)$. Notice that by (108) $\langle B(G),\sigma(B(G),\mathcal{N})\rangle^* = \mathcal{N}$.

Let G be a $\mathbb{K}-$normed space, then the strong operator topology $\tau_{st}(G)$ on $B(G)$ is defined to be the locally convex topology generated by the following set of seminorms $\{q_v : B(G) \ni A \mapsto \|Av\|_G \mid v \in G\}$. Hence $\tau_{st}(G)$ is a Hausdorff topology, in fact a base of the neighbourhoods of $A \in B(G)$ is the class of the sets $U_{\overline{v},\varepsilon}(A) \doteqdot \{B \in B(G) \mid \sup_{k=1,\dots,n} \|(A-B)\overline{v}_k\|_G < \varepsilon\}$, with $\overline{v}$ running in $\bigcup_{n\in\mathbb{N}} G^n$ and ϵ in $\mathbb{R}^+ - \{0\}$. So $B \in \overline{\{\mathbf{0}\}}$, the closure of $\{\mathbf{0}\}$ in the strong operator topology, if and only if $\|Bv\|_G < \varepsilon$, for all $\varepsilon \in \mathbb{R}^+ - \{0\}, v \in G$, that is $B = \mathbf{0}$. Hence $\overline{\{\mathbf{0}\}} = \{\mathbf{0}\}$ and then $\tau_{st}(G)$ is of Hausdorff. By Ch. 6, §1, $n°3$, of [**INT**]

$$\mathcal{N}_{st}(G) \doteqdot \langle B(G),\tau_{st}(G)\rangle^* = \mathfrak{L}_{\mathbb{K}}(\{\psi_{(\phi,v)} \mid (\phi,v) \in G^* \times G\}). \tag{111}$$

Here

$$\psi_{(\phi,v)} : B(G) \ni T \mapsto \phi(Tv) \in \mathbb{K}.$$

Here if Z is a $\mathbb{K}-$linear space and $S \subseteq Z$ then $\mathfrak{L}_{\mathbb{K}}(S)$ is the space of all $\mathbb{K}-$linear combinations of elements in S.

The first locally convex space in which are mainly interested, is $\langle B(G),\sigma(B(G),\mathcal{N}_{st}(G))\rangle$, for which by (108) we have

$$\langle B(G),\sigma(B(G),\mathcal{N}_{st}(G))\rangle^* = \mathcal{N}_{st}(G). \tag{112}$$

Notice that by what said $\sigma(B(G),\mathcal{N}_{st}(G))$ is the topology on $B(G)$ generated by the set of seminorms associated to the set $\{\psi_{(\phi,v)} \mid (\phi,v) \in G^*\times G\}$, hence $\sigma(B(G),\mathcal{N}_{st}(G))$ is nothing but the usual weak operator topology on $B(G)$.

Notice that by (107), and the Hahn-Banach theorem applied to G we have that $\sigma(B(G),\mathcal{N}_{st}(G))$ is a topology of Hausdorff.

Let G be a complex Hilbert space. We define

$$\mathcal{N}_{pd}(G) \doteqdot B(G)_*.$$

Here $B(G)_*$ is the "predual" of the von Neumann algebra $B(G)$, see for example Definition $2.4.17.$ of [**BR**], or Definition $2.13.$, Ch. 2 of [**Tak**], So every $\omega \in \mathcal{N}_{pd}(G)$ has the following form, see Proposition $2.4.6$ of [**BR**] or statement $(ii.4)$ of Theorem $2.6.$, Ch. 2 of [**Tak**]

$$\omega : B(G) \ni B \mapsto \sum_{n=0}^{\infty} \langle u_n, Bw_n\rangle \in \mathbb{C}. \tag{113}$$

Here $\{u_n\}_{n\in\mathbb{N}}, \{w_n\}_{n\in\mathbb{N}} \subset G$ are such that $\sum_{n=0}^{\infty} \|u_n\|^2 < \infty$ and $\sum_{n=0}^{\infty} \|w_n\|^2 < \infty$.

We say that ω *is determined by* $\{u_n\}_{n\in\mathbb{N}}, \{w_n\}_{n\in\mathbb{N}}$ if (113) holds. Notice that ω is well-defined, indeed for all $B \in B(G)$ we have $\sum_{n=0}^{\infty} |\langle u_n, Bw_n\rangle|^2 \leq$

$\|B\|^2\left(\sum_{n=0}^{\infty}\|u_n\|^2\right)\left(\sum_{n=0}^{\infty}\|w_n\|^2\right) < \infty$, hence there exists $\omega(B)$ and $\omega \in B(G)^*$, so

$$\mathcal{N}_{pd}(G) \subseteq B(G)^*. \tag{114}$$

The second locally convex space in which are mainly interested is $\langle B(G), \sigma(B(G), \mathcal{N}_{pd}(G))\rangle$, for which by (108) we have

$$\langle B(G), \sigma(B(G), \mathcal{N}_{pd}(G))\rangle^* = \mathcal{N}_{pd}(G). \tag{115}$$

By the fact that every $\omega \in \mathcal{N}_{st}(G)$ is determined by the $\{u_n\}_{n=1}^N,\ \{w_n\}_{n=1}^N$, for some $N \in \mathbb{N}$, we have that $\mathcal{N}_{st}(G) \subset \mathcal{N}_{pd}(G)$. Hence being $\sigma(B(G), \mathcal{N}_{st}(G))$ a topology of Hausdorff we can conclude by (107) that it is so also the $\sigma(B(G), \mathcal{N}_{pd}(G))-$ topology.

Notice that by what said $\sigma(B(G), \mathcal{N}_{pd}(G))$ is the topology on $B(G)$ generated by the set of seminorms associated to the set $\mathcal{N}_{pd}(G)$, hence is nothing but the usual sigma-weak operator topology on $B(G)$, see for example for its definition Section $2.4.1$ of [**BR**], so often we shall refer to it just as the sigma-weak operator topology on $B(G)$.

We want just to remark that as a corollary of the beforementioned invariance property for the weak-integration, when we change the topology τ on Y with any other Hausdorff topology compatible with it, we deduce by (111) that $f : X \to B(G)$ is scalarly essentially $(\mu, B(G))-$integrable with respect to the measure μ and with respect to the $\sigma(B(G), \mathcal{N}_{st}(G))$ topology on $B(G)$, if and only if it is so with respect to the strong topology $\tau_{st}(G)$ on $B(G)$, and in this case their weak-integrals coincide. Let $\mathcal{A}$ be a $\mathbb{K}-$Banach algebra then for all $A, B \in \mathcal{A}$ set $[A, B] \doteqdot AB - BA$, while the map $\mathcal{R} : \mathcal{A} \to B(\mathcal{A})$ and $\mathcal{L} : \mathcal{A} \to B(\mathcal{A})$, have been defined in (7). Let G be a $\mathbb{K}-$Banach space and $\mathcal{N} \subseteq B(G)^*$ a linear subspace of the normed space $B(G)^*$, then we introduce the following notations

$$\begin{cases} \mathcal{N}^* \subseteq B(G) \overset{def}{\Leftrightarrow} (\exists\, Y_0 \subseteq B(G))(\mathcal{N}^* = \{\hat{A} \upharpoonright \mathcal{N} \mid A \in Y_0\}); \\ \mathcal{N}^* \overset{\|\cdot\|}{\subseteq} B(G) \overset{def}{\Leftrightarrow} \\ (\exists\, Y_0 \subseteq B(G))(\forall \phi \in \mathcal{N}^*)(\exists\, A \in Y_0)((\phi = \hat{A} \upharpoonright \mathcal{N}) \wedge (\|\phi\|_{\mathcal{N}^*} = \|A\|_{B(G)})). \end{cases}$$

Here $(\hat{\cdot}) : B(G) \to (B(G)^*)^*$ is the canonical isometric embedding of $B(G)$ into its bidual.

By statement (iii) of Theorem $2.6.$, Ch. 2 of [**Tak**], or Proposition $2.4.18$ of [**BR**]

$$\mathcal{N}_{pd}(G)^* \overset{\|\cdot\|}{\subseteq} B(G). \tag{116}$$

Let $\mathbf{H} : \mathcal{B}_Y \to \Pr(G)$ be a spectral measure in G on $\mathcal{B}_Y$ then we continue to follow the notation

$$(\forall \sigma \in \mathcal{B}(\mathbb{C}))(G_\sigma \doteqdot \mathbf{H}(\sigma)G),$$

without expressing the dependence on $\mathbf{H}$ everywhere it does not cause confusion.

In this Chapter we assume to be fixed a complex Banach space G, a locally compact space X a complex Radon measure μ on X, a possibly unbounded scalar type spectral operator R with spectrum $\sigma(R)$ and resolution of the the identity E.

For each map $f : U \subset \mathbb{C} \to \mathbb{C}$ we denote by $\widetilde{f}$ the $\mathbf{0}-$extension of f to $\mathbb{C}$.

Finally we shall denote with $\mathfrak{F}_{ess}(X;\mu)$ the seminormed space, with the seminorm $\|\cdot\|_{\mathfrak{F}_{ess}(X;\mu)}$, of all maps $H : X \to \mathbb{C}$ such that

$$\|H\|_{\mathfrak{F}_{ess}(X;\mu)} \doteqdot \int^{\bullet} |H(x)|\ d|\mu|(x) < \infty.$$

By $\mu - l.a.e.(X)$ we shall mean "locally almost everywhere on X with respect to μ". Moreover if $f : X_0 \to \mathbb{C}$ is a map defined $\mu - l.a.e.(X)$, then we convene to say that $f \in$

$\mathfrak{F}_{ess}(X;\mu)$ if there exists a map $F : X \to \mathbb{C}$ such that $F \upharpoonright X_0 = f$ and $F \in \mathfrak{F}_{ess}(X;\mu)$. In such a case we set

$$\|f\|_{\mathfrak{F}_{ess}(X;\mu)} \doteqdot \|F\|_{\mathfrak{F}_{ess}(X;\mu)}. \tag{117}$$

(117) is well-defined since the definition is independent of which map $F \in \mathfrak{F}_{ess}(X;\mu)$ extends f, as an application of statement (a) of Proposition 1, $n^{\circ}1$, $\S1$, Ch. V of **[INT]**. Moreover let $\langle Y,\tau\rangle$ be a locally convex space and $f : X_0 \to Y$ a map defined $\mu - l.a.e.(X)$, then we for brevity say that the map $f : X \to \langle Y,\tau\rangle$ is scalarly essentially $(\mu, Y)-$integrable if there exists a map $F : X \to Y$ such that $F \upharpoonright X_0 = f$ and $F : X \to \langle Y,\tau\rangle$ is scalarly essentially $(\mu, Y)-$integrable. In this case we define

$$\int f(x)\,d\mu(x) \doteqdot \int F(x)\,d\mu(x). \tag{118}$$

This definition is well-defined since it does not depend by the scalarly essentially $(\mu, Y)-$integrable map F which extends f. Indeed let for all $k \in \{1,2\}$ the map $F_k : X \to Y$ be such that $F_k \upharpoonright X_0 = f$ and $F_k : X \to \langle Y,\tau\rangle$ be scalarly essentially $(\mu, Y)-$integrable, then for all $\omega \in \langle Y,\tau\rangle^*, k \in \{1,2\}$

$$\omega\left(\int F_k(x)\,d\mu(x)\right) = \int \omega(F_k(x))\,d\mu(x) = \int \chi_{X_0}(x)\omega(F_k(x))\,d\mu(x).$$

Next for all $\forall x \in X$, $\chi_{X_0}(x)\omega(F_1(x)) = \chi_{X_0}(x)\omega(f(x)) = \chi_{X_0}(x)\omega(F_2(x))$, so for all $\omega \in \langle Y,\tau\rangle^*$

$$\omega\left(\int F_1(x)\,d\mu(x)\right) = \omega\left(\int F_2(x)\,d\mu(x)\right)$$

then by (107) follows $\int F_1(x)\,d\mu(x) = \int F_2(x)\,d\mu(x)$.

Now we will show some result about which functions are scalarly essentially $(\mu, B(G))-$integrable with respect to the $\sigma(B(G),\mathcal{N})-$topology. Here $\mathcal{N} \subseteq B(G)^*$, such that separates the points of $B(G)$ and $\mathcal{N}^* \subseteq B(G)$. Then we apply these results to the case when G is a Hilbert space and $\mathcal{N} = \mathcal{N}_{pd}(G)$.

THEOREM 2.2. *Let G be a complex Banach space, a subspace $\mathcal{N} \subseteq B(G)^*$ be such that $\mathcal{N}$ separates the points of $B(G)$ and*

$$\mathcal{N}^* \subseteq B(G).$$

Let $F : X \to B(G)$ be a map such that for all $\omega \in \mathcal{N}$ the map $\omega \circ F : X \to \mathbb{C}$ is $\mu-$measurable and

$$(X \ni x \mapsto \|F(x)\|_{B(G)}) \in \mathfrak{F}_{ess}(X;\mu). \tag{119}$$

Then the map $F : X \to \langle B(G), \sigma(B(G),\mathcal{N})\rangle$ is scalarly essentially $(\mu, B(G))-$integrable, if in addition $\mathcal{N}^* \overset{\|\cdot\|}{\subseteq} B(G)$ then its weak-integral is such that

$$\left\|\int F(x)\,d\mu(x)\right\|_{B(G)} \leq \int^{\bullet} \|F(x)\|_{B(G)}\,d|\mu|(x). \tag{120}$$

PROOF. For all $\omega \in \mathcal{N}$ we have $|\omega(F(x))| \leq \|\omega\|\|F(x)\|_{B(G)}$, hence for all $\omega \in \mathcal{N}$

$$\int^{\bullet} |\omega(F(x))|\,d|\mu|(x) \leq \|\omega\| \int^{\bullet} \|F(x)\|_{B(G)}\,d|\mu|(x). \tag{121}$$

Moreover the map $\omega \circ F$ is $\mu-$measurable by hypothesis, therefore by (121) and Proposition 9, $\S1$, $n^{\circ}3$, Ch. 5 of **[INT]** we have that $\omega \circ F$ is essentially $\mu-$integrable.

By this fact we can define the following map

$$\Psi:\mathcal{N}\ni\omega\mapsto\int\omega(F(x))\,d\mu(x)\in\mathbb{C}$$

which is linear. Moreover for any essentially $\mu-$integrable map $H:X\to\mathbb{C}$

$$\left|\int H(x)\,d\mu(x)\right|\leq\int^{\bullet}|H(x)|\,d|\mu|(x), \tag{122}$$

hence by (121)

$$\Psi\in\mathcal{N}^{*}. \tag{123}$$

Finally by the duality property $\mathcal{N}^{*}\subseteq B(G)$ in hypothesis the statement follows by (123) and (121). $\square$

REMARK 2.3. Let G be a complex Hilbert space, then the statement of Theorem 2.2 holds if we set $\mathcal{N}\doteqdot\mathcal{N}_{pd}(G)$. Indeed we have the duality property (116).

Now we give similar results for $\mathcal{N}=\mathcal{N}_{st}(G)$.

LEMMA 2.4. *Let G be reflexive, that is $(G^{*})^{*}$ is isometric to G through the natural injective embedding of any normed space into its bidual. In addition let $B:G^{*}\times G\to\mathbb{C}$ be a bounded bilinear form, that is*

$$(\exists\, C>0)(\forall(\phi,v)\in G^{*}\times G)(|B(\phi,v)|\leq C\|\phi\|_{G^{*}}\|v\|_{G}).$$

Then

$$(\exists!L\in B(G))(\forall\phi\in G^{*})(\forall v\in G)(B(\phi,v)=\phi(L(v))$$

and $\|L\|_{B(G)}\leq\|B\|$, where $\|B\|\doteqdot\sup_{\{(\phi,v)\mid\|\phi\|_{G^{*}},\|v\|_{G}\leq 1\}}|B(\phi,v)|$.

PROOF. For all $v\in G$ let $T(v):G^{*}\ni\phi\mapsto B(\phi,v)\in\mathbb{C}$ so $T(v)\in(G^{*})^{*}$ such that $\|T(v)\|_{(G^{*})^{*}}\leq\|B\|\cdot\|v\|_{G}$. G is reflexive, hence $(\forall v\in G)(\exists!L(v)\in G)(\forall\phi\in G^{*})(\phi(L(v))=T(v)(\phi))$, in addition $\|L(v)\|_{G}=\|T(v)\|_{(G^{*})^{*}}\leq\|B\|\cdot\|v\|_{G}$. L is linear by the linearity of T and by the fact that G^{*} separates the points of G by the Hahn-Banach theorem. Thus L is linear and bounded and $\|L\|_{B(G)}\leq\|B\|$. This implies the existence of L. Let now $L'\in B(G)$ be another operator with the same property, so for all $\phi\in G^{*},v\in G$, $\phi(L(v))=\phi(L'(v))$, thus by the Hahn-Banach theorem for all $v\in G$ $L(v)=L'(v)$, which shows the uniqueness. $\square$

THEOREM 2.5. *Let G be reflexive, $F:X\to B(G)$ be a map such that for all $(\phi,v)\in G^{*}\times G$ the map $X\ni x\mapsto\phi(F(x)v)\in\mathbb{C}$ is $\mu-$measurable, finally assume that* (119) *holds. Then the map $F:X\to\langle B(G),\sigma(B(G),\mathcal{N}_{st}(G))\rangle$ is scalarly essentially $(\mu,B(G))-$integrable and its weak-integral satisfies* (120).

PROOF. We have for all $\phi\in G^{*},v\in G,x\in X$ that $|\phi(F(x)v)|\leq\|\phi\|\|v\|\|F(x)\|_{B(G)}$, hence

$$\int^{\bullet}|\phi(F(x)v)|\,d|\mu|(x)\leq\|\phi\|\|v\|\int^{\bullet}\|F(x)\|_{B(G)}\,d|\mu|(x) \tag{124}$$

Furthermore the map $X\ni x\mapsto\phi(F(x)v)$ is $\mu-$measurable by hypothesis, therefore by (124) and Proposition 9, $\S 1$, $n^{\circ}3$, Ch. 5 of [**INT**] we have that $X\ni x\mapsto\phi(F(x)v)$ is essentially $\mu-$integrable.

By this fact we can define the following map

$$B:G^{*}\times G\ni(\phi,v)\mapsto\int\phi(F(x)v)\,d\mu(x)\in\mathbb{C},$$

which is bilinear. So by (122) and (124)

$$|B(\phi, v)| \leq \|\phi\|\|v\| \int^{\bullet} \|F(x)\|_{B(G)}\, d|\mu|(x).$$

Hence B is a bounded bilinear form whose norm is such that $\|B\| \leq \int^{\bullet} \|F(x)\|_{B(G)}\, d|\mu|(x)$, then the statement by Lemma 2.4. $\square$

COROLLARY 2.6. *Let G be reflexive, $F : X \to B(G)$ a map $\sigma(B(G), \mathcal{N}_{st}(G))-$continuous, i.e. for all $(\phi, v) \in G^* \times G$ the map $X \ni x \mapsto \phi(F(x)v) \in \mathbb{C}$ is continuous, finally assume that* (119) *holds.*

Then the map $F : X \to \langle B(G), \sigma(B(G), \mathcal{N}_{st}(G))\rangle$ is scalarly essentially $(\mu, B(G))-$integrable and its weak-integral satisfies (120).

PROOF. By definition of $\mu-$measurability we have that the continuity condition implies that for all $(\phi, v) \in G^* \times G$ the map $X \ni x \mapsto \phi(F(x)v) \in \mathbb{C}$ is $\mu-$measurable, hence the statement by Theorem 2.5. $\square$

3. Commutation and restriction properties

Let $\mathbf{H} : \mathcal{B}_Y \to \Pr(G)$ be a spectral measure in G on $\mathcal{B}_Y$, then in the sequel we shall introduce a special class of subspaces of $B(G)^*$, the class of all "$\mathbf{H}-$appropriate sets", which allows one to show two important properties for proving the main Extension Theorem 2.25. These are

(1) "Commutation" property: Theorem 2.13, for a general $E-$appropriate set $\mathcal{N}$, and Corollary 2.14 for $\mathcal{N} = \mathcal{N}_{pd}(G)$ or $\mathcal{N} = \mathcal{N}_{st}(G)$;
(2) "Restriction" property: Theorem 2.22 for a general $E-$appropriate set $\mathcal{N}$.

LEMMA 2.7. *Let $A \in B(G)$ such that $AR \subseteq RA$ and $f \in Bor(\sigma(R))$. Then*

$$Af(R) \subseteq f(R)A.$$

PROOF. By Corollary 18.2.4. of **[DS]**

$$(\forall \sigma \in \mathcal{B}(\mathbb{C}))([A, E(\sigma)] = \mathbf{0}). \tag{125}$$

By (8) for all $T \in B(G))$, $\mathcal{R}(T), \mathcal{L}(T) \in B(B(G)))$, so by using the notations in Preliminaries 1.1, we have for all $n \in \mathbb{N}$

$$\begin{aligned}
\mathbf{I}_{\mathbb{C}}^{E}(f_n)A &= \left(\mathcal{L}(A) \circ \mathbf{I}_{\mathbb{C}}^{E}\right)(f_n) \\
&= \mathbf{I}_{\mathbb{C}}^{\mathcal{L}(A)\circ E}(f_n) && \text{by (31), } \mathcal{L}(A) \in B(B(G)) \\
&= \mathbf{I}_{\mathbb{C}}^{\mathcal{R}(A)\circ E}(f_n) && \text{by (125)} \\
&= \left(\mathcal{R}(A) \circ \mathbf{I}_{\mathbb{C}}^{E}\right)(f_n) && \text{by (31), } \mathcal{R}(A) \in B(B(G)) \\
&= A\,\mathbf{I}_{\mathbb{C}}^{E}(f_n). && (126)
\end{aligned}$$

Let $x \in Dom(f(R))$ then by (35), the fact that $A \in B(G)$ and (126)

$$Af(R)x = \lim_{n\to\infty} \mathbf{I}_{\mathbb{C}}^{E}(f_n)Ax.$$

Hence (35) implies $Ax \in Dom(f(R))$ and

$$f(R)Ax = \lim_{n\to\infty} \mathbf{I}_{\mathbb{C}}^{E}(f_n)Ax = Af(R)x.$$

$\square$

LEMMA 2.8. *Let $\mathcal{N} \subseteq B(G)^*$ be such that $\sigma(B(G),\mathcal{N})$ is a Hausdorff topology, $A \in B(G)$, and the map $X \ni x \mapsto f_x \in Bor(\sigma(R))$ be such that $\widetilde{f_x} \in \mathfrak{L}_E^\infty(\sigma(R))$ $\mu - l.a.e.(X)$. Assume that*

(1) *the map $X \ni x \mapsto f_x(R) \in \langle B(G), \sigma(B(G),\mathcal{N})\rangle$ is scalarly essentially $(\mu, B(G))-$integrable;*
(2) *$\phi \circ \mathcal{R}(A) \in \mathcal{N}$ and $\phi \circ \mathcal{L}(A) \in \mathcal{N}$, for all $\phi \in \mathcal{N}$;*
(3) *$AR \subseteq RA$.*

Then

$$\left[\int f_x(R)\, d\mu(x),\ A\right] = \mathbf{0}.$$

PROOF. By the hypothesis $\widetilde{f_x} \in \mathfrak{L}_E^\infty(\sigma(R)),\ \mu - l.a.e.(X)$ and statement (c) of Theorem 18.2.11. of [**DS**] applied to the scalar type spectral operator R, we have $f_x(R) \in B(G)$, $\mu - l.a.e.(X)$. Let us set $X_0 \doteqdot \{x \in X \mid f_x(R) \in B(G)\}$. By the hypothesis (1) we deduce that there is $F : X \to B(G)$ such that

- $(\forall x \in X_0)(F(x) = f_x(R))$;
- $F : X \to \langle B(G), \sigma(B(G),\mathcal{N})\rangle$ is scalarly essentially $(\mu, B(G))-$integrable.

Thus by definition

$$\int f_x(R)\, d\mu(x) \doteqdot \int F(x)\, d\mu(x) \tag{127}$$

Notice that for all $x \in X, \phi \in \mathcal{N}$

$$\chi_{X_0}(x)\, \phi \circ \mathcal{L}(A)(F(x)) = \chi_{X_0}(x)\, \phi \circ \mathcal{R}(A)(F(x)), \tag{128}$$

since by Lemma 2.7 for all $x \in X_0$

$$F(x)A = f_x(R)A = Af_x(R) = AF(x).$$

Moreover for all $\phi \in \mathcal{N}$

$$\begin{cases} \int \phi \circ \mathcal{L}(A)\,(F(x))\ d\mu(x) = \int \chi_{X_0}(x)\, \phi \circ \mathcal{L}(A)\,(F(x))\ d\mu(x), \\ \int \phi \circ \mathcal{R}(A)\,(F(x))\ d\mu(x) = \int \chi_{X_0}(x)\, \phi \circ \mathcal{R}(A)\,(F(x))\ d\mu(x). \end{cases} \tag{129}$$

Indeed $\phi \circ \mathcal{L}(A) \in \mathcal{N}$ hence $X \ni x \mapsto \phi \circ \mathcal{L}(A)\,(F(x))$ is essentially $\mu-$integrable so by Proposition 9 $n^\circ 3$ §1 Ch 5 of [**INT**]

$$\int^\bullet \left|\chi_{X_0}(x)\, \phi \circ \mathcal{L}(A)\,(F(x))\right| d\,|\mu|(x) \leq \int^\bullet \left|\phi \circ \mathcal{L}(A)\,(F(x))\right| d\,|\mu|(x) < \infty.$$

Furthermore by Proposition 6 $n^\circ 2$ §5 Ch 4 of [**INT**] $X \ni x \mapsto \chi_{X_0}(x)\, \phi \circ \mathcal{L}(A)\,(F(x))$ is $\mu-$measurable. Thus by Proposition 9 $n^\circ 3$ §1 Ch 5 of [**INT**] the map $X \ni x \mapsto \chi_{X_0}(x)\, \phi \circ \mathcal{L}(A)\,(F(x))$ is essentially $\mu-$integrable and we obtain the first statement of (129) by the fact that two essentially $\mu-$integrable maps that are equal $\mu - l.a.e.(X)$ have the same integral. In the same way it is possible to show also the second statement of (129).

Therefore for all $\phi \in \mathcal{N}$

$$\begin{aligned}
\phi\left(\int f_x(R)\, d\mu(x) A\right) &= \phi \circ \mathcal{L}(A)\left(\int f_x(R)\, d\mu(x)\right) \\
&= \phi \circ \mathcal{L}(A)\left(\int F(x)\, d\mu(x)\right) && \text{by (127)} \\
&= \int \phi \circ \mathcal{L}(A)\,(F(x))\; d\mu(x) && \text{by } \phi \circ \mathcal{L}(A) \in \mathcal{N} \\
&= \int \phi \circ \mathcal{R}(A)\,(F(x))\; d\mu(x) && \text{by (129), (128)} \\
&= \phi \circ \mathcal{R}(A)\left(\int F(x)\, d\mu(x)\right) && \text{by } \phi \circ \mathcal{R}(A) \in \mathcal{N} \\
&= \phi\left(A \int f_x(R)\, d\mu(x)\right). && \text{by (127)}
\end{aligned}$$

Then the statement by (107) $\square$

REMARK 2.9. By definition of $\mathcal{N}_{st}(G)$, see (111), the hypothesis (2) of Lemma 2.8 holds for all $A \in B(G)$ and for $\mathcal{N} = \mathcal{N}_{st}(G)$. Moreover $\sigma(B(G), \mathcal{N}_{st}(G))$ is a Hausdorff topology on $B(G)$.

Let G be a Hilbert space, by (113) we note that for all $A \in B(G)$ we have $\omega \circ \mathcal{L}(A) \in \mathcal{N}_{pd}(G)$, and $\omega \circ \mathcal{R}(A) \in \mathcal{N}_{pd}(G)$, indeed if ω is determined by $\{u_n\}_{n\in\mathbb{N}}, \{w_n\}_{n\in\mathbb{N}}$, then $\omega \circ \mathcal{L}(A)$, (respectively $\omega \circ \mathcal{R}(A)$), is determined by $\{u_n\}_{n\in\mathbb{N}}, \{Aw_n\}_{n\in\mathbb{N}}$, (respectively $\{A^* u_n\}_{n\in\mathbb{N}}, \{w_n\}_{n\in\mathbb{N}}$). Hence the hypothesis (2) of Lemma 2.8 holds for all $A \in B(G)$ and for $\mathcal{N} = \mathcal{N}_{pd}(G)$. Furthermore $\sigma(B(G), \mathcal{N}_{pd}(G))$ is a Hausdorff topology on $B(G)$.

REMARK 2.10. By Definition $18.2.1$ of [**DS**] for all $\sigma \in \mathcal{B}(\mathbb{C})$, $E(\sigma)R \subseteq RE(\sigma)$, thus hypothesis (3) of Lemma 2.8 holds for $A \doteqdot E(\sigma)$.

DEFINITION 2.11 (**H−appropriate set**). Let $\mathbf{H} : \mathcal{B}_Y \to \Pr(G)$ be a spectral measure in G on $\mathcal{B}_Y$, see Preliminaries 1.1. Then we define $\mathcal{N}$ to be an **H−appropriate set** , if

(1) $\mathcal{N} \subseteq B(G)^*$ linear subspace;
(2) $\mathcal{N}$ separates the points of $B(G)$, namely

$$(\forall T \in B(G))(T \neq \mathbf{0} \Rightarrow (\exists\, \phi \in \mathcal{N})(\phi(T) \neq 0));$$

(3) for all $\phi \in \mathcal{N}, \sigma \in \mathcal{B}_Y$

$$\phi \circ \mathcal{R}(\mathbf{H}(\sigma)) \in \mathcal{N} \quad \phi \circ \mathcal{L}(\mathbf{H}(\sigma)) \in \mathcal{N}. \tag{130}$$

Furthermore $\mathcal{N}$ is an **H−appropriate set with the duality property** if $\mathcal{N}$ is an $\mathbf{H}-$appropriate set such that

$$\mathcal{N}^* \subseteq B(G).$$

Finally $\mathcal{N}$ is an $\mathbf{H}-$appropriate set with the isometric duality property if $\mathcal{N}$ is an $\mathbf{H}-$appropriate set such that

$$\mathcal{N}^* \overset{\|\cdot\|}{\subseteq} B(G).$$

REMARK 2.12. Some comments about the previous definition. The separation property is equivalent to require that $\sigma(B(G), \mathcal{N})$ is a Hausdorff topology on $B(G)$, while (130) is equivalent to require that for all $\sigma \in \mathcal{B}_Y$ the maps on $B(G)$, $\mathcal{R}(\mathbf{H}(\sigma))$ and $\mathcal{L}(\mathbf{H}(\sigma))$ are continuous with respect to the $\sigma(B(G), \mathcal{N})-$ topology. Moreover the duality property

$\mathcal{N}^* \subseteq B(G)$ ensures that suitable scalarly essentially $\mu-$integrable maps with respect to the $\sigma(B(G),\mathcal{N})-$ topology, are $(\mu, B(G))-$integrable, see Theorem 2.2.

Finally by Remark 2.9 $\mathcal{N}_{st}(G)$ and $\mathcal{N}_{pd}(G)$, in case in which G is a Hilbert space, are $\mathbf{H}-$appropriate sets for any spectral measure $\mathbf{H}$, furthermore by (116), $\mathcal{N}_{pd}(G)$ is an $\mathbf{H}-$appropriate set with the isometric duality property.

THEOREM 2.13 (**Commutation 1**). *Let $\mathcal{N}$ be an $E-$appropriate set, the map $X \ni x \mapsto f_x \in Bor(\sigma(R))$ be such that $\widetilde{f}_x \in \mathfrak{L}_E^\infty(\sigma(R))$ $\mu - l.a.e.(X)$. Assume that the map $X \ni x \mapsto f_x(R) \in \langle B(G), \sigma(B(G),\mathcal{N})\rangle$ is scalarly essentially $(\mu, B(G))-$integrable. Then for all $\sigma \in \mathcal{B}(\mathbb{C})$*

$$\left[\int f_x(R)\, d\mu(x),\ E(\sigma)\right] = \mathbf{0}. \tag{131}$$

PROOF. $\mathcal{N}$ being an $E-$appropriate set ensures that hypothesis (2) of Lemma 2.8 is satisfied for $A \doteqdot E(\sigma)$ for all $\sigma \in \mathcal{B}(\mathbb{C})$, so the statement by Remark 2.10 and Lemma 2.8. □

COROLLARY 2.14 (**Commutation 2**). (131) *holds if we replace $\mathcal{N}$ in Theorem 2.13 with $\mathcal{N}_{st}(G)$ or with $\mathcal{N}_{pd}(G)$ and assume that G is a Hilbert space.*

PROOF. By Remark 2.12 and Theorem 2.13. □

Now we are going to present some results necessary for showing the Restriction property in Theorem 2.22, namely that the map $X \ni x \mapsto f_x(R_\sigma \upharpoonright G_\sigma) \in \langle B(G_\sigma), \sigma(B(G_\sigma), \mathcal{N}_\sigma)\rangle$ is scalarly essentially $(\mu, B(G_\sigma))-$integrable, where $\mathcal{N}$ is a $E-$appropriate set, and $\mathcal{N}_\sigma$ could be thought as the "restriction" of $\mathcal{N}$ to $B(G_\sigma)$ for all $\sigma \in \mathcal{B}(\mathbb{C})$.

In particular when $\mathcal{N} = \mathcal{N}_{st}(G)$, respectively $\mathcal{N} = \mathcal{N}_{pd}(G)$, we can replace $\mathcal{N}_\sigma$ with $\mathcal{N}_{st}(G_\sigma)$, respectively $\mathcal{N}_{pd}(G_\sigma)$, Proposition 2.23.

LEMMA 2.15. *Let $\mathbf{H} : \mathcal{B}_Y \to \mathrm{Pr}(G)$ be a spectral measure in G on $\mathcal{B}_Y$, see Preliminaries 1.1. Then for all $\sigma \in \mathcal{B}_Y$ $G = G_\sigma \bigoplus G_{\sigma'}$, where $\sigma' \doteqdot \complement\sigma$.*

PROOF. $\mathbf{H}(\sigma) + \mathbf{H}(\sigma') = \mathbf{H}(\sigma \cup \sigma') = \mathbf{1}$ so $\mathbf{H}(\sigma') = \mathbf{1} - \mathbf{H}(\sigma)$ and $\mathbf{H}(\sigma)\mathbf{H}(\sigma') = \mathbf{H}(\sigma')\mathbf{H}(\sigma) = \mathbf{0}$. Hence for all $v \in G$, $v = \mathbf{H}(\sigma)v + \mathbf{H}(\sigma')v$, or $G = G_\sigma + G_{\sigma'}$. But for any $\delta \in \mathcal{B}_Y$ we have $G_\delta = \{y \in G \mid y = \mathbf{H}(\delta)y\}$ then $G_\sigma \cap G_{\sigma'} = \{y \in G \mid y = \mathbf{H}(\sigma)\mathbf{H}(\sigma')y\} = \{\mathbf{0}\}$. Thus $G_\sigma + G_{\sigma'} = G_\sigma \bigoplus G_{\sigma'}$. □

DEFINITION 2.16. Let $\mathbf{H} : \mathcal{B}_Y \to \mathrm{Pr}(G)$ be a spectral measure in G on $\mathcal{B}_Y$, $\sigma \in \mathcal{B}_Y$ and $\sigma' \doteqdot \complement\sigma$. Then Lemma 2.15 allows us to define the operator $\xi_\sigma^{\mathbf{H}} : B(G_\sigma) \to B(G)$, such that for all $T_\sigma \in B(G_\sigma)$

$$\xi_\sigma^{\mathbf{H}}(T_\sigma) \doteqdot T_\sigma \oplus \mathbf{0}_{\sigma'} \in B(G). \tag{132}$$

Whenever it does not cause confusion we shall denote $\xi_\sigma^{\mathbf{H}}$ simply by ξ_σ. Here $\mathbf{0}_{\sigma'} \in B(G_{\sigma'})$ is the null element of the space $B(G_{\sigma'})$, while the direct sum of two operators $T_\sigma \in B(G_\sigma)$ and $T_{\sigma'} \in B(G_{\sigma'})$ is the following standard definition

$$(T_\sigma \oplus T_{\sigma'}) : G_\sigma \bigoplus G_{\sigma'} \ni (v_\sigma \oplus v_{\sigma'}) \mapsto T_\sigma v_\sigma \oplus T_{\sigma'} v_{\sigma'} \in G_\sigma \bigoplus G_{\sigma'}.$$

LEMMA 2.17. *Let $\mathbf{H} : \mathcal{B}_Y \to \mathrm{Pr}(G)$ be a spectral measure in G on $\mathcal{B}_Y$, then for all $\forall \sigma \in \mathcal{B}_Y, T_\sigma \in B(G_\sigma)$ we have that*

$$\xi_\sigma^{\mathbf{H}}(T_\sigma) = T_\sigma \mathbf{H}(\sigma). \tag{133}$$

Hence $\xi_\sigma^{\mathbf{H}}$ is well-defined, injective, $\xi_\sigma^{\mathbf{H}} \in B(B(G_\sigma), B(G))$ and $\|\xi_\sigma^{\mathbf{H}}\|_{B(B(G_\sigma),B(G))} \leq \|\mathbf{H}(\sigma)\|_{B(G)}$.

PROOF. Let $\sigma \in \mathcal{B}_Y$ then for all $v \in G$ we have

$$(T_\sigma \oplus \mathbf{0}_{\sigma'})v = (T_\sigma \oplus \mathbf{0}_{\sigma'})(\mathbf{H}(\sigma)v \oplus \mathbf{H}(\sigma')v) = (T_\sigma \mathbf{H}(\sigma)v \oplus \mathbf{0}) = T_\sigma \mathbf{H}(\sigma)v,$$

then the first part. Let $T_\sigma \in B(G_\sigma)$ such that $\xi_\sigma(T_\sigma) = \mathbf{0}$, then $T_\sigma \mathbf{H}(\sigma) = \mathbf{0}$, which implies that for all $v_\sigma \in G_\sigma$ we have $T_\sigma v_\sigma = T_\sigma \mathbf{H}(\sigma) v_\sigma = \mathbf{0}$. So $T_\sigma = \mathbf{0}_\sigma$. Let us consider $\mathbf{H}(\sigma) \in B(G, G_\sigma)$, and $T_\sigma \in B(G_\sigma, G)$, so $T_\sigma \mathbf{H}(\sigma) \in B(G)$ and $\|T_\sigma \mathbf{H}(\sigma)\|_{B(G)} \leq \|T_\sigma\|_{B(G_\sigma,G)} \cdot \|\mathbf{H}(\sigma)\|_{B(G,G_\sigma)} = \|T_\sigma\|_{B(G_\sigma)} \cdot \|\mathbf{H}(\sigma)\|_{B(G)}$. □

Notice that by (133) and the fact that $B(G_\sigma)$ is a Banach space, it is possible to show that $\xi_\sigma(B(G_\sigma))$ is a Banach subspace of $B(G)$, thus ξ_σ has a continuous inverse.

REMARK 2.18. Let $\mathbf{H} : \mathcal{B}_Y \to \Pr(G)$ be a spectral measure in G on $\mathcal{B}_Y$, and $\sigma \in \mathcal{B}_Y$. If we consider the product space $G_\sigma \times G_{\sigma'}$ with the standard linearization and define

$$\begin{cases} \|(x_\sigma, x_{\sigma'})\|_\oplus \doteqdot \|x_\sigma + x_{\sigma'}\|_G, \\ I : G_\sigma \times G_{\sigma'} \ni (x_\sigma, x_{\sigma'}) \mapsto x_\sigma + x_{\sigma'} \in G, \end{cases} \tag{134}$$

then by $G = G_\sigma \bigoplus G_{\sigma'}$, see Lemma 2.15, the two spaces $\langle G_\sigma \times G_{\sigma'}, \|\cdot\|_\oplus \rangle$ and $\langle G, \|\cdot\|_G \rangle$ are isomorphics, thus isometric and I is an isometry between them. It is not difficult to see that the topology induced by the norm $\|\cdot\|_\oplus$ is the product topology on $G_\sigma \times G_{\sigma'}$ [2], which implies the following property that in any case we prefer to show directly.

PROPOSITION 2.19. *Let $\mathbf{H} : \mathcal{B}_Y \to \Pr(G)$ be a spectral measure in G on $\mathcal{B}_Y$ and assume the notations in* (134) *and Definition 2.16. For all $T_\sigma \in B(G_\sigma)$ and $T_{\sigma'} \in B(G_{\sigma'})$ set*

$$T_\sigma \times T_{\sigma'} : G_\sigma \times G_{\sigma'} \ni (x_\sigma, x_{\sigma'}) \mapsto (T_\sigma x_\sigma, T_{\sigma'} x_{\sigma'}) \in G_\sigma \times G_{\sigma'}$$

Then

$$\begin{cases} T_\sigma \oplus T_{\sigma'} = I(T_\sigma \times T_{\sigma'})I^{-1} = T_\sigma \mathbf{H}(\sigma) + T_{\sigma'} \mathbf{H}(\sigma') \in B(G) \\ T_\sigma \times T_{\sigma'} = I^{-1}(T_\sigma \mathbf{H}(\sigma) + T_{\sigma'} \mathbf{H}(\sigma'))I \in B(G_\sigma \times G_{\sigma'}). \end{cases} \tag{135}$$

PROOF. $I(T_\sigma \times T_{\sigma'})I^{-1}(x_\sigma \oplus x_{\sigma'}) = I(T_\sigma x_\sigma, T_{\sigma'} x_{\sigma'}) = T_\sigma x_\sigma \oplus T_{\sigma'} x_{\sigma'}$, for all $x_\sigma \in G_\sigma$ and $x_{\sigma'} \in G_{\sigma'}$, so the first equality. For all $x \in G$

$$\begin{aligned} I(T_\sigma \times T_{\sigma'})I^{-1}(x) &= I(T_\sigma \times T_{\sigma'})I^{-1}(\mathbf{H}(\sigma)x + \mathbf{H}(\sigma')x) \\ &= I(T_\sigma \mathbf{H}(\sigma)x, T_{\sigma'} \mathbf{H}(\sigma')x) \\ &= T_\sigma \mathbf{H}(\sigma)x + T_{\sigma'} \mathbf{H}(\sigma')x. \end{aligned}$$

Then the second equality. The third equality is by the second and the fact that I is an isometry. □

Notice that by the first statement in (135) we obtain (133).

DEFINITION 2.20. Let $\mathbf{H} : \mathcal{B}_Y \to \Pr(G)$ be a spectral measure in G on $\mathcal{B}_Y$ and $\mathcal{N} \subseteq B(G)^*$. We define for all $\sigma \in \mathcal{B}_Y, \psi \in \mathcal{N}$

$$\begin{cases} \psi_\sigma^{\mathbf{H}} \doteqdot \psi \circ \xi_\sigma^{\mathbf{H}} \in B(G_\sigma)^* \\ \mathcal{N}_\sigma^{\mathbf{H}} \doteqdot \{\psi_\sigma^{\mathbf{H}} \mid \psi \in \mathcal{N}\}, \end{cases} \tag{136}$$

---

[2] Indeed let $\sigma \in \mathcal{B}_Y$ such that $\mathbf{H}(\sigma) \neq \mathbf{0}$, set $M \doteqdot \max\{\|\mathbf{H}(\sigma)\|, \|\mathbf{H}(\sigma')\|\}$ and for all $r > 0$ define $B_r^\oplus(\mathbf{0}) \doteqdot \{(x_\sigma, x_{\sigma'}) \in G_\sigma \times G_{\sigma'} \mid \|(x_\sigma, x_{\sigma'})\|_\oplus < r\}$. Thus for all $\varepsilon > 0$ by setting $\eta \doteqdot \frac{\varepsilon}{2}$ we have $B_\eta(\mathbf{0}_\sigma) \times B_\eta(\mathbf{0}'_\sigma) \subset B_\varepsilon^\oplus(\mathbf{0})$, while for all $\varepsilon_1, \varepsilon_2 > 0$ by setting $\zeta \doteqdot \frac{\min\{\varepsilon_1, \varepsilon_2\}}{M}$ we have $B_\zeta^\oplus(\mathbf{0}) \subset B_{\varepsilon_1}(\mathbf{0}_\sigma) \times B_{\varepsilon_2}(\mathbf{0}'_\sigma)$.

where $\xi_\sigma^{\mathbf{H}}$ has been defined in (132). We shall express $\psi_\sigma^{\mathbf{H}}$ and $\mathcal{N}_\sigma^{\mathbf{H}}$ simply by ψ_σ and $\mathcal{N}_\sigma$ respectively, whenever it does not cause confusion.

PROPOSITION 2.21. *Let $\mathbf{H} : \mathcal{B}_Y \to \Pr(G)$ be a spectral measure in G on $\mathcal{B}_Y$, $\mathcal{N} \subseteq B(G)^*$ such that $\mathcal{N}$ separates the points of $B(G)$ and $\sigma \in \mathcal{B}_Y$. Then $\mathcal{N}_\sigma$ separates the points of $B(G_\sigma)$.*

PROOF. Let $T_\sigma \in B(G_\sigma) - \{\mathbf{0}_\sigma\}$, by Lemma 2.17 ξ_σ is injective so $\xi_\sigma(T_\sigma) \neq \mathbf{0}$. But $\mathcal{N}$ separates the points of $B(G)$, so there is $\psi \in \mathcal{N}$ such that $\psi(\xi_\sigma(T_\sigma)) \neq 0$. □

THEOREM 2.22 (**Restriction**). *Let $\mathcal{N}$ be an $E-$appropriate set, the map $X \ni x \mapsto f_x \in Bor(\sigma(R))$ be such that $\widetilde{f_x} \in \mathfrak{L}_E^\infty(\sigma(R)) \quad \mu - l.a.e.(X)$ Assume that the map $X \ni x \mapsto f_x(R) \in \langle B(G), \sigma(B(G),\mathcal{N})\rangle$ is scalarly essentially $(\mu, B(G))-$integrable. Then for all $\sigma \in \mathcal{B}(\mathbb{C})$ the map $X \ni x \mapsto f_x(R_\sigma \upharpoonright G_\sigma) \in \langle B(G_\sigma), \sigma(B(G_\sigma),\mathcal{N}_\sigma)\rangle$ is scalarly essentially $(\mu, B(G_\sigma))-$integrable and*

$$\int f_x(R_\sigma \upharpoonright G_\sigma)\, d\mu(x) = \int f_x(R)\, d\mu(x) \upharpoonright G_\sigma. \tag{137}$$

PROOF. Let $\sigma \in \mathcal{B}(\mathbb{C})$ then (43) implies that for all $x \in X$ the operator $f_x(R_\sigma \upharpoonright G_\sigma)$ is well-defined. By the hypothesis $\widetilde{f_x} \in \mathfrak{L}_E^\infty(\sigma(R))$, $\mu - l.a.e.(X)$ and statement (c) of Theorem 18.2.11. of [**DS**] applied to the scalar type spectral operator R, we have $f_x(R) \in B(G)$, $\mu - l.a.e.(X)$. Let us set

$$X_0 \doteqdot \{x \in X \mid f_x(R) \in B(G)\},$$

thus by statement (2) of Lemma 1.7 we obtain

$$(\forall x \in X_0)(f_x(R_\sigma \upharpoonright G_\sigma) \in B(G_\sigma)). \tag{138}$$

Hence $f_x(R_\sigma \upharpoonright G_\sigma) \in B(G_\sigma)$, $\mu - l.a.e.(X)$. So by Proposition 2.21 and (107) it is well-defined the statement that $X \ni x \mapsto f_x(R_\sigma \upharpoonright G_\sigma) \in \langle B(G_\sigma), \sigma(B(G_\sigma),\mathcal{N}_\sigma)\rangle$ is scalarly essentially $(\mu, B(G_\sigma))-$integrable. By hypothesis we deduce that there is $F : X \to B(G)$ such that

- $(\forall x \in X_0)(F(x) = f_x(R))$;
- $F : X \to \langle B(G), \sigma(B(G),\mathcal{N})\rangle$ is scalarly essentially $(\mu, B(G))-$integrable.

Thus by (118)

$$\int f_x(R)\, d\mu(x) \doteqdot \int F(x)\, d\mu(x) \tag{139}$$

Now for all $\sigma \in \mathcal{B}(\mathbb{C})$ let us define the map $F^\sigma : X \to B(G_\sigma)$ such that for all $x \in X$

$$F^\sigma(x) \doteqdot E(\sigma)F(x) \upharpoonright G_\sigma.$$

By (138) we can claim that

(1) $(\forall x \in X_0)(F^\sigma(x) = f_x(R_\sigma \upharpoonright G_\sigma))$;
(2) the map $F^\sigma : X \to \langle B(G_\sigma), \sigma(B(G_\sigma),\mathcal{N}_\sigma)\rangle$ is scalarly essentially $(\mu, B(G_\sigma))-$integrable, moreover

$$\int F^\sigma(x)\, d\mu(x) = \int f_x(R)\, d\mu(x) \upharpoonright G_\sigma. \tag{140}$$

Then the statement will follow by setting according (118)

$$\int f_x(R_\sigma \upharpoonright G_\sigma)\, d\mu(x) \doteqdot \int F^\sigma(x)\, d\mu(x).$$

For all $x \in X_0$

$$\begin{aligned} F^{\sigma}(x) &= E(\sigma) f_x(R) \upharpoonright G_{\sigma} \\ &= f_x(R) E(\sigma) \upharpoonright G_{\sigma} \text{ by } [f_x(R), E(\sigma)] = \mathbf{0} \\ &= f_x(R_{\sigma} \upharpoonright G_{\sigma}) \qquad \text{by Key Lemma 1.7.} \end{aligned}$$

Hence (1) of our claim follows. For all $\psi \in \mathcal{N}, x \in X$

$$\begin{aligned} \psi \circ \mathcal{L}(E(\sigma)) \circ \mathcal{R}(E(\sigma)) \, (F(x)) &\doteq \psi \left(E(\sigma) F(x) E(\sigma) \right) \\ &= \psi_{\sigma} \left(E(\sigma) F(x) \upharpoonright G_{\sigma} \right) \\ &\doteq \psi_{\sigma} \left(F^{\sigma}(x) \right). \end{aligned} \tag{141}$$

Here in the second equality we deduce by Lemma 2.17 that for all $T \in B(G)$ we have $\xi_{\sigma} \left(E(\sigma) T \upharpoonright G_{\sigma} \right) = E(\sigma) T E(\sigma)$. $F : X \to \langle B(G), \sigma(B(G), \mathcal{N}) \rangle$ is scalarly essentially $\mu-$integrable, and for all $\psi \in \mathcal{N}$, $\psi \circ \mathcal{L}(E(\sigma)) \circ \mathcal{R}(E(\sigma)) \in \mathcal{N}$, hence by (141) the map

$$F^{\sigma} : X \to \langle B(G_{\sigma}), \sigma(B(G_{\sigma}), \mathcal{N}_{\sigma}) \rangle \text{ is scalarly essentially } \mu-\text{integrable.}$$

Now by (131) we have for all $v \in G_{\sigma}$

$$\int f_x(R) \, d\mu(x) v = \int f_x(R) \, d\mu(x) E(\sigma) v = E(\sigma) \int f_x(R) \, d\mu(x) v \in G_{\sigma}, \tag{142}$$

moreover $\int f_x(R) \, d\mu(x) \in B(G)$ so

$$\int f_x(R) \, d\mu(x) \upharpoonright G_{\sigma} \in B(G_{\sigma}).$$

Therefore for all $\psi \in \mathcal{N}$

$$\begin{aligned} &\psi_{\sigma} \left(\int f_x(R) \, d\mu(x) \upharpoonright G_{\sigma} \right) \\ &= \psi_{\sigma} \left(E(\sigma) \int f_x(R) \, d\mu(x) \upharpoonright G_{\sigma} \right) \quad \text{by (142)} \\ &= \psi \left(E(\sigma) \int f_x(R) \, d\mu(x) \, E(\sigma) \right) \quad \text{by Lemma 2.17} \\ &\doteq \psi \circ \mathcal{L}(E(\sigma)) \circ \mathcal{R}(E(\sigma)) \left(\int f_x(R) \, d\mu(x) \right) \\ &\doteq \psi \circ \mathcal{L}(E(\sigma)) \circ \mathcal{R}(E(\sigma)) \left(\int F(x) \, d\mu(x) \right) \quad \text{by (139)} \\ &= \int \psi \circ \mathcal{L}(E(\sigma)) \circ \mathcal{R}(E(\sigma)) \, (F(x)) \; d\mu(x) \quad \text{by } \psi \circ \mathcal{L}(E(\sigma)) \circ \mathcal{R}(E(\sigma)) \in \mathcal{N} \\ &= \int \psi_{\sigma} \left(F^{\sigma}(x) \right) \; d\mu(x) \quad \text{by (141).} \end{aligned}$$

Hence (140) by (109) and (110) and the statement follows. ☐

PROPOSITION 2.23. *For all* $\sigma \in \mathcal{B}(\mathbb{C})$

$$(\mathcal{N}_{st}(G))_{\sigma} = \mathcal{N}_{st}(G_{\sigma}) \textit{ and } (\mathcal{N}_{pd}(G))_{\sigma} = \mathcal{N}_{pd}(G_{\sigma}); \tag{143}$$

PROOF. By the Hahn-Banach theorem

$$(G_{\sigma})^{*} = \{ \phi \upharpoonright G_{\sigma} \mid \phi \in G^{*} \}. \tag{144}$$

Then we have

$$\begin{aligned}(\mathcal{N}_{st}(G))_\sigma &\doteq \mathfrak{L}_{\mathbb{C}}(\{\psi_{(\phi,v)}\circ\xi_\sigma \mid (\phi,v)\in G^*\times G\})\\ &= \mathfrak{L}_{\mathbb{C}}(\{\psi_{(\phi\upharpoonright G_\sigma,w)} \mid (\phi,w)\in G^*\times G_\sigma\})\\ &= \mathfrak{L}_{\mathbb{C}}(\{\psi_{(\rho,w)} \mid (\rho,w)\in (G_\sigma)^*\times G_\sigma\})\\ &\doteq \mathcal{N}_{st}(G_\sigma).\end{aligned}$$

Here in the third equality we used (144), while in the second equality we considered that for all $(\phi,v)\in G^*\times G$ and for all $T_\sigma\in B(G_\sigma)$

$$\begin{aligned}\psi_{(\phi,v)}\circ\xi_\sigma(T_\sigma) &= \phi(T_\sigma E(\sigma)v) && \text{by (133)}\\ &= (\phi\upharpoonright G_\sigma)\,(T_\sigma E(\sigma)v)\\ &= \psi_{(\phi\upharpoonright G_\sigma, E(\sigma)v)}(T_\sigma).\end{aligned}\tag{145}$$

Let G be a complex Hilbert space then

$$(\mathcal{N}_{pd}(G))_\sigma =$$

$$\left\{\left(\sum_{n=0}^\infty \psi_{(u_n^\dagger,w_n)}\right)\circ\xi_\sigma \,\middle|\, \{u_n\}_{n\in\mathbb{N}},\ \{w_n\}_{n\in\mathbb{N}}\subset G, \sum_{n=0}^\infty\|u_n\|_G^2<\infty, \sum_{n=0}^\infty\|w_n\|_G^2<\infty\right\} =$$

$$\left\{\sum_{n=0}^\infty (\psi_{(u_n^\dagger,w_n)}\circ\xi_\sigma) \,\middle|\, \{u_n\}_{n\in\mathbb{N}},\ \{w_n\}_{n\in\mathbb{N}}\subset G, \sum_{n=0}^\infty\|u_n\|_G^2<\infty, \sum_{n=0}^\infty\|w_n\|_G^2<\infty\right\} =$$

$$\left\{\sum_{n=0}^\infty \psi_{(u_n^\dagger\upharpoonright G_\sigma, E(\sigma)w_n)} \,\middle|\, \{u_n\}_{n\in\mathbb{N}},\ \{w_n\}_{n\in\mathbb{N}}\subset G, \sum_{n=0}^\infty\|u_n\|_G^2<\infty, \sum_{n=0}^\infty\|w_n\|_G^2<\infty\right\} =$$

$$\left\{\sum_{n=0}^\infty \psi_{(E(\sigma)^*u_n)^\dagger\upharpoonright G_\sigma, E(\sigma)w_n)} \,\middle|\, \{u_n\}_{n\in\mathbb{N}},\ \{w_n\}_{n\in\mathbb{N}}\subset G, \sum_{n=0}^\infty\|u_n\|_G^2, \sum_{n=0}^\infty\|w_n\|_G^2<\infty\right\} =$$

$$\left\{\sum_{n=0}^\infty \psi_{(a_n^\dagger,b_n)} \,\middle|\, \{a_n\}_{n\in\mathbb{N}},\ \{b_n\}_{n\in\mathbb{N}}\subset G_\sigma, \sum_{n=0}^\infty\|a_n\|_{G_\sigma}^2<\infty, \sum_{n=0}^\infty\|b_n\|_{G_\sigma}^2<\infty\right\} =$$

$$\mathcal{N}_{pd}(G_\sigma)$$

Here for any Hilbert space F we set $u^\dagger\in F^*$ such that $u^\dagger(v)\doteqdot\langle u,v\rangle$ for all $u,v\in F$, and the series in the first equality is converging with respect to the strong operator topology on $B(G)^*$, while all the others are converging with respect to the strong operator topology on $B(G_\sigma)^*$.

The first equality follows by 113, the third is by (145), the forth by the fact that $E(\sigma)\upharpoonright G_\sigma=\mathbf{1}_\sigma$ the identity operator on G_σ. Now we shall show the fifth equality. Notice that

$$\sum_{n=0}^\infty\|E(\sigma)w_n\|_{G_\sigma}^2 \doteq \sum_{n=0}^\infty\|E(\sigma)w_n\|_G^2 \le \|E(\sigma)\|^2\sum_{n=0}^\infty\|w_n\|_G^2<\infty.$$

While by the fact that $\dagger : H\to H^*$ is a semilinear isometry, we have for all $n\in\mathbb{N}$ that exists only one $a_n\in G_\sigma$ such that $a_n^\dagger=(E(\sigma)^*u_n)^\dagger\upharpoonright G_\sigma$ moreover

$$\|a_n\|_{G_\sigma}=\|\,(E(\sigma)^*u_n)^\dagger\upharpoonright G_\sigma\|_{G_\sigma^*}.$$

Next

$$\begin{aligned}\|\,(E(\sigma)^{*}u_n)^{\dagger}\restriction G_\sigma\|_{G_\sigma^*} &= \sup_{\{v\in G_\sigma \mid \|v\|_{G_\sigma}\leq 1\}} |\,\langle E(\sigma)^{*}u_n,v\rangle\,| \\ &= \sup_{\{v\in G_\sigma \mid \|v\|_{G_\sigma}\leq 1\}} |\,\langle u_n,v\rangle\,| \leq \sup_{\{v\in G \mid \|v\|_{G}\leq 1\}} |\,\langle u_n,v\rangle\,| \\ &= \|u_n^{\dagger}\|_{G^*} = \|u_n\|_G.\end{aligned}$$

Hence $\sum_{n=0}^{\infty}\|a_n\|_{G_\sigma}^2 \leq \sum_{n=0}^{\infty}\|u_n\|_G^2 < \infty$ and the fifth equality follows. $\square$

4. Extension theorem for integral equalities with respect to the $\sigma(B(G),\mathcal{N})-$ topology

In the present section wil shall prove the Extension Theorems for integration with respect to the $\sigma(B(G),\mathcal{N})-$topology, when $\mathcal{N}$ is an $E-$appropriate set: Theorems 2.25 and when $\mathcal{N}$ is an $E-$appropriate set with the duality property: Corollary 2.26. As an application we shall consider the cases of the sigma-weak topology: Corollary 2.28 and Corollary 2.29; and weak operator topology: Corollary 2.27, and Corollary 2.30. In this section it will be adopted all the notations defined in Section 2.

THEOREM 2.24. *Let $\mathcal{N}$ be an $E-$appropriate set and $\{\sigma_n\}_{n\in\mathbb{N}}$ be an $E-$sequence (see Definition 1.10) and the map $X \ni x \mapsto f_x \in Bor(\sigma(R))$ be such that $\widetilde{f_x} \in \mathfrak{L}_E^{\infty}(\sigma(R))$ $\mu - l.a.e.(X)$. Let $X \ni x \mapsto f_x(R) \in \langle B(G),\sigma(B(G),\mathcal{N})\rangle$ be scalarly essentially $(\mu,B(G))-$integrable and $g,h\in Bor(\sigma(R))$.*

If for all $n\in\mathbb{N}$

$$g(R_{\sigma_n}\restriction G_{\sigma_n})\int f_x(R_{\sigma_n}\restriction G_{\sigma_n})\,d\,\mu(x)\subseteq h(R_{\sigma_n}\restriction G_{\sigma_n}) \tag{146}$$

then

$$g(R)\int f_x(R)\,d\,\mu(x)\restriction\Theta = h(R)\restriction\Theta. \tag{147}$$

In (146) *the weak-integral is with respect to the measure μ and with respect to the $\sigma(B(G_{\sigma_n}),\mathcal{N}_{\sigma_n})-$ topology, while in* (147)

$$\Theta \doteqdot Dom\left(g(R)\int f_x(R)\,d\,\mu(x)\right)\cap Dom(h(R))$$

and the weak-integral is with respect to the measure μ and with respect to the $\sigma(B(G),\mathcal{N})-$ topology.

PROOF. (146) is well set since Theorem 2.22.

By (47) for all $y\in\Theta$

$$g(R)\int f_x(R)\,d\,\mu(x)\,y = \lim_{n\in\mathbb{N}} E(\sigma_n)g(R)\int f_x(R)\,d\,\mu(x)\,y$$

by statement (g) of Theorem 18.2.11 of [**DS**] and (131)

$$= \lim_{n\in\mathbb{N}} g(R)\int f_x(R)\,d\,\mu(x)\,E(\sigma_n)y$$

by (137) and Lemma 1.7 applied to $g(R)$

$$= \lim_{n\in\mathbb{N}} g(R_{\sigma_n}\restriction G_{\sigma_n})\int f_x(R_{\sigma_n}\restriction G_{\sigma_n})\,d\,\mu(x)\,E(\sigma_n)y$$

by hypothesis (146)

$$= \lim_{n\in\mathbb{N}} h(R_{\sigma_n} \restriction G_{\sigma_n})\, E(\sigma_n) y$$

by Lemma 1.7 and statement (g) of Theorem 18.2.11 of [**DS**]

$$= \lim_{n\in\mathbb{N}} E(\sigma_n) h(R)\, y$$

$$= h(R)\, y. \tag{148}$$

In the last equality we considered (47). □

THEOREM 2.25 ($\sigma(B(G),\mathcal{N})-$ **Extension Theorem**). *Let R be a possibly unbounded scalar type spectral operator in G, E its resolution of the identity, $\mathcal{N}$ an $E-$appropriate set. Let the map $X \ni x \mapsto f_x \in Bor(\sigma(R))$ be such that $\widetilde{f_x} \in \mathfrak{L}_E^\infty(\sigma(R))$ $\mu - l.a.e.(X)$ and the map $X \ni x \mapsto f_x(R) \in \langle B(G), \sigma(B(G),\mathcal{N})\rangle$ be scalarly essentially $(\mu, B(G))-$integrable. Finally let $g, h \in Bor(\sigma(R))$ and $\widetilde{h} \in \mathfrak{L}_E^\infty(\sigma(R))$.*

If $\{\sigma_n\}_{n\in\mathbb{N}}$ is an $E-$sequence and for all $n \in \mathbb{N}$

$$g(R_{\sigma_n} \restriction G_{\sigma_n}) \int f_x(R_{\sigma_n} \restriction G_{\sigma_n})\, d\,\mu(x) \subseteq h(R_{\sigma_n} \restriction G_{\sigma_n}) \tag{149}$$

then $h(R) \in B(G)$ and

$$g(R) \int f_x(R)\, d\,\mu(x) = h(R). \tag{150}$$

In (149) *the weak-integral is with respect to the measure μ and with respect to the $\sigma(B(G_{\sigma_n}), \mathcal{N}_{\sigma_n})-$ topology, while in* (150) *the weak-integral is with respect to the measure μ and with respect to the $\sigma(B(G),\mathcal{N})-$ topology.*

Notice that $g(R)$ is a possibly **unbounded** operator in G.

PROOF. Theorem 18.2.11. of [**DS**] and hypothesis $\widetilde{h} \in \mathfrak{L}_E^\infty(\sigma(R))$ imply that $h(R) \in B(G)$, so by (147) we can deduce

$$g(R) \int f_x(R)\, d\,\mu(x) \subseteq h(R). \tag{151}$$

Let us set

$$(\forall n \in \mathbb{N})(\delta_n \doteqdot \overset{-1}{|g|}([0,n])). \tag{152}$$

We claim that

$$\begin{cases} \bigcup_{n\in\mathbb{N}} \delta_n = \sigma(R) \\ n \geq m \Rightarrow \delta_n \supseteq \delta_m \\ (\forall n \in \mathbb{N})(g(\delta_n) \text{ is bounded. }) \end{cases} \tag{153}$$

Since $|g| \in Bor(\sigma(R))$ we have $\delta_n \in \mathcal{B}(\mathbb{C})$ for all $n \in \mathbb{N}$, so $\{\delta_n\}_{n\in\mathbb{N}}$ is an $E-$sequence, hence by (47)

$$\lim_{n\in\mathbb{N}} E(\delta_n) = \mathbf{1} \tag{154}$$

with respect to the strong operator topology on $B(G)$.

Indeed the first equality follows by $\bigcup_{n\in\mathbb{N}}\delta_n \doteq \bigcup_{n\in\mathbb{N}}\overset{-1}{|g|}([0,n]) = \overset{-1}{|g|}\left(\bigcup_{n\in\mathbb{N}}[0,n]\right) = \overset{-1}{|g|}(\mathbb{R}^+) = Dom(g) \doteqdot \sigma(R)$, the second by the fact that $\overset{-1}{|g|}$ preserves the inclusion, the third by the inclusion $|g|(\delta_n) \subseteq [0,n]$. Hence our claim. By the third statement of (153), $\delta_n \in \mathcal{B}(\mathbb{C})$ and statement 3 of Lemma 1.7

$$(\forall n\in\mathbb{N})(E(\delta_n)G \subseteq Dom(g(R))). \tag{155}$$

By (131) and (155) for all $n\in\mathbb{N}$

$$\int f_x(R)\,d\,\mu(x)E(\delta_n)G \subseteq E(\delta_n)G \subseteq Dom(g(R)).$$

Therefore

$$(\forall n\in\mathbb{N})(\forall v\in G)\left(E(\delta_n)v \in Dom\left(g(R)\int f_x(R)\,d\,\mu(x)\right)\right).$$

Hence by (154)

$$\mathbf{D} \doteqdot Dom\left(g(R)\int f_x(R)\,d\,\mu(x)\right) \text{ is dense in } G. \tag{156}$$

But $\int f_x(R)\,d\,\mu(x) \in B(G)$ and $g(R)$ is closed by Theorem $18.2.11.$ of [**DS**], so by Lemma 1.15 we have

$$g(R)\int f_x(R)\,d\,\mu(x) \text{ is closed.} \tag{157}$$

But we know that $h(R)\in B(G)$ so by (151) we deduce

$$g(R)\int f_x(R)\,d\,\mu(x) \in B(\mathbf{D},G). \tag{158}$$

The (157), (158) and Lemma 1.16 allow us to state that $\mathbf{D}$ is closed in G, therefore by (156)

$$\mathbf{D} = G.$$

Hence by (151) we can conclude that the statement holds. □

Now we shall show a corollary of the previous theorem, in which we give conditions on the maps f_x ensuring that $f_x(R)\in B(G)$, and that $X \ni x \mapsto f_x(R) \in B(G)$ is scalarly essentially $(\mu, B(G))-$integrable with respect to the $\sigma(B(G),\mathcal{N})-$ topology.

COROLLARY 2.26 ($\sigma(B(G),\mathcal{N})-$ **Extension Theorem. Duality case.**). *Let $\mathcal{N}$ be an $E-$appropriate set with the duality property and $X \ni x \mapsto f_x \in Bor(\sigma(R))$. Assume that there is $X_0 \subseteq X$ such that $\complement X_0$ is $\mu-$locally negligible and $\widetilde{f_x} \in \mathfrak{L}_E^{\infty}(\sigma(R))$ for all $x\in X_0$, moreover let there exist $F : X \to B(G)$ extending $X_0 \ni x \mapsto f_x(R) \in B(G)$ such that for all $\omega\in\mathcal{N}$ the map $X \ni x \mapsto \omega(F(x)) \in \mathbb{C}$ is $\mu-$measurable and*

$$(X \ni x \mapsto \|F(x)\|_{B(G)}) \in \mathfrak{F}_{ess}(X;\mu). \tag{159}$$

If $g,h \in Bor(\sigma(R))$ such that $\widetilde{h} \in \mathfrak{L}_E^{\infty}(\sigma(R))$ and $\{\sigma_n\}_{n\in\mathbb{N}}$ is an $E-$sequence such that for all $n\in\mathbb{N}$ holds (149) *then the statement of Theorem 2.25 holds. Moreover if $\mathcal{N}$ is an $E-$appropriate set with the isometric duality property*

$$\left\|\int f_x(R)\,d\mu(x)\right\|_{B(G)} \leq \int^{\bullet} \|f_x(R)\|_{B(G)}\,d|\mu|(x).$$

PROOF. By the duality property of hypothesis, and Theorem 2.2 the map $X \ni x \mapsto f_x(R) \in \langle B(G), \sigma(B(G),\mathcal{N})\rangle$ is scalarly essentially $(\mu, B(G))-$integrable. Hence the first part of the statement by Theorem 2.25. The inequality follows by (120), (118) and (117). $\square$

Now we will give the corollaries of the previous two results in the cases in which $\mathcal{N} = \mathcal{N}_{st}(G)$ or $\mathcal{N} = \mathcal{N}_{pd}(G)$ and G be a Hilbert space.

COROLLARY 2.27. *The statement of Theorem 2.25 holds if $\mathcal{N}$ is replaced by $\mathcal{N}_{st}(G)$ and $\mathcal{N}_{\sigma_n}$ is replaced by $\mathcal{N}_{st}(G_{\sigma_n})$, for all $n \in \mathbb{N}$.*

PROOF. By Remark 2.12 we know that $\mathcal{N}_{st}(G)$ is an $E-$appropriate set, therefore the statement by (143) and Theorem 2.25. $\square$

COROLLARY 2.28. *The statement of Theorem 2.25 holds if G is a complex Hilbert space, $\mathcal{N}$ is replaced by $\mathcal{N}_{pd}(G)$ and $\mathcal{N}_{\sigma_n}$ is replaced by $\mathcal{N}_{pd}(G_{\sigma_n})$, for all $n \in \mathbb{N}$.*

PROOF. By Remark 2.12 we know that $\mathcal{N}_{pd}(G)$ is in particular an $E-$appropriate set, therefore the statement by (143) and Theorem 2.25. $\square$

THEOREM 2.29 (**Sigma-weak Extension Theorem**). *Let G be a Hilbert space, then the statement of Corollary 2.26 holds if we set $\mathcal{N} \doteqdot \mathcal{N}_{pd}(G)$ and $\mathcal{N}_{\sigma_n} \doteqdot \mathcal{N}_{pd}(G_{\sigma_n})$ for all $n \in \mathbb{N}$.*

PROOF. By Remark 2.12 $\mathcal{N}_{pd}(G)$ is an $E-$appropriate set with the isometric duality property, so we obtain the statement by Corollary 2.26 and by (143). $\square$

COROLLARY 2.30 (**Weak Extension Theorem**). *Let G be reflexive, then the statement of Corollary 2.26 holds if we set $\mathcal{N} \doteqdot \mathcal{N}_{st}(G)$ and $\mathcal{N}_{\sigma_n} \doteqdot \mathcal{N}_{st}(G_{\sigma_n})$ for all $n \in \mathbb{N}$.*

PROOF. By Theorem 2.5 we have that the map $X \ni x \mapsto f_x(R) \in \langle B(G), \sigma(B(G), \mathcal{N}_{st}(G))\rangle$ is scalarly essentially $(\mu, B(G))-$integrable. Hence the first part of the statement by Corollary 2.27. While the inequality follows by (120), (118) and (117). $\square$

REMARK 2.31. In the case in which G is an Hilbert space we can obtain Corollary 2.30 as an application of the duality property of the predual $\mathcal{N}_{pd}(G)$. Indeed as we know $\mathcal{N}_{st}(G) \subset \mathcal{N}_{pd}(G)$, hence by the Hahn-Banach theorem for all $\Psi_0 \in \mathcal{N}_{st}(G)^*$ there exists $\Psi \in \mathcal{N}_{pd}(G)^*$ such that $\Psi \upharpoonright \mathcal{N}_{st}(G) = \Psi_0$, thus by the duality property $\mathcal{N}_{pd}(G)^* = B(G)$ we obtain $(\forall \Psi_0 \in \mathcal{N}_{st}(G)^*)(\exists B \in B(G))(\forall \omega \in \mathcal{N}_{st}(G))(\Psi_0(\omega) = \omega(B))$, which ensures that the weak-integral with respect to the measure μ and with respect to the weak operator topology of the map $X \ni x \mapsto f_x(R) \in B(G)$ belongs to $B(G)$.

REMARK 2.32. Let $\mathsf{D} \subset G$ be a linear subspace of G and $E : \mathcal{B}(\mathbb{C}) \to \Pr(G)$ be a countably additive spectral measure, then by (31) for all $f \in \mathbf{TM}$, $\phi \in G^*$ and $v \in \mathsf{D}$ that

$$\left|\phi\left(\mathbf{I}_{\mathbb{C}}^{E}(f)v\right)\right| = \left|\int f(\lambda)\, d\left(\psi_{\phi,v} \circ E\right)(\lambda)\right| \leq 4M\|f\|_{\sup}\|\phi\|\|v\|, \tag{160}$$

where $M \doteqdot \sup_{\delta\in\mathcal{B}(\mathbb{C})} \|E(\delta)\|$, $\psi_{\phi,v} : B(G) \ni A \mapsto \phi(Av) \in \mathbb{C}$ and $\mathbf{TM}$ is the space of all totally $\mathcal{B}(\mathbb{C})-$measurable complex maps on $\mathbb{C}$. Next we know that

$$H(\mathbb{C}) \subset \mathbf{TM}. \tag{161}$$

Here $H(\mathbb{C})$ is the space of all compactly supported complex continuous functions on $\mathbb{C}$, with the direct limit topology, of the spaces $H(\mathbb{C}; K)$ with K running in the class of all

compact subsets of $\mathbb{C}$; where $H(\mathbb{C};K)$ is the space of all complex continuous functions $f:\mathbb{C}\to\mathbb{C}$ such that $\mathrm{supp}(f)\subset K$ with the topology of uniform convergence. Let us set

$$F_w^{\mathsf{D}} \doteqdot \overline{B(\mathsf{D},G)} \text{ in } \mathfrak{L}_w(\mathsf{D},G),$$

where $\mathfrak{L}_w(\mathsf{D},G)$ is the Hausdorff locally convex space of all linear operators on D at values in G with the topology generated by the following set of seminorms

$$\{\mathfrak{L}_w(\mathsf{D},G)\ni B\mapsto |q_{\phi,v}(B)| \mid (\phi,v)\in G^*\times\mathsf{D}\},$$

where $q_{\phi,v}(B)\doteqdot\phi(Bv)$ for all $(\phi,v)\in G^*\times\mathsf{D}$ and $B\in\mathfrak{L}_w(\mathsf{D},G)$, while $B(\mathsf{D},G)$ is the space of all bounded operators belonging to $\mathfrak{L}_w(\mathsf{D},G)$. By (161) we can define

$$\mathbf{m}_E : H(\mathbb{C})\ni f\mapsto \left(\mathbf{I}_{\mathbb{C}}^E(f)\upharpoonright\mathsf{D}\right)\in F_w^{\mathsf{D}}$$

Moreover by (160) we have, with the notations in 1.9, that for all compact K the operator $\mathbf{m}_E\upharpoonright H(\mathbb{C};K)$ is continuous. Therefore as a corollary of the general result in statement (ii) Proposition 5, $n^\circ 4$, $\S 4$, Ch 2 of [**TVS**] about locally convex final topologies, so in particular for the inductive limit topology, we deduce that $\mathbf{m}_E$ is continuous on $H(\mathbb{C})$ i.e.

$$\mathbf{m}_E \text{ is a vector measure on } \mathbb{C} \text{ with vales in } F_w^{\mathsf{D}}.$$

Here, by generalizing to the complex case the definition 1, $n^\circ 1$, $\S 2$, Ch 6 of [**INT**], we call a vector measure on a locally compact space X with values in a complex Hausdorff locally convex space Y any $\mathbb{C}-$linear continuous map $\mathbf{m}:H(X)\to Y$. Furthermore for all $(\phi,v)\in G^*\times\mathsf{D}$

$$\begin{aligned} q_{\phi,v}\circ\mathbf{m}_E &= \psi_{\phi,v}\circ\mathbf{I}_{\mathbb{C}}^E\upharpoonright H(\mathbb{C}) \\ &= \mathbf{I}_{\mathbb{C}}^{\psi_{\phi,v}\circ E}\upharpoonright H(\mathbb{C}). \quad \text{by (31)} \end{aligned}$$

Hence

$$\mathfrak{L}_1(\mathbb{C};q_{\phi,v}\circ\mathbf{m}_E)=\mathfrak{L}_1(\mathbb{C};\psi_{\phi,v}\circ E),$$

where the left hand side it is intended in the sense of Ch 4 of [**INT**], while the right hand side it is intended in the standard sense, and for all $f\in\mathfrak{L}_1(\mathbb{C};q_{\phi,v}\circ\mathbf{m}_E)$

$$\int f(\lambda)\,d\,(q_{\phi,v}\circ\mathbf{m}_E)(\lambda)=\int f(\lambda)\,d\,(\psi_{\phi,v}\circ E)(\lambda) \tag{162}$$

Finally let us assume that D is dense, then for all $f\in Bor(\mathrm{supp}\,E)$ such that $Dom(f(E))=\mathsf{D}$ by (35) we have

$$f(E)\in F_w^{\mathsf{D}},$$

and by Theorem $18.2.11$ of [**DS**] for all $(\phi,v)\in G^*\times\mathsf{D}$ we have $f\in\mathfrak{L}_1(\mathbb{C};\psi_{\phi,v}\circ E)$ and

$$q_{\phi,v}(f(E))=\int f(\lambda)\,d\,(\psi_{\phi,v}\circ E)(\lambda). \tag{163}$$

Therefore by adopting the definitions in $n^\circ 2$, $\S 2$, Ch 6 of [**INT**], we deduce by (162) that each $f\in Bor(\mathrm{supp}\,E)$ such that $Dom(f(E))=\mathsf{D}$ is *essentially integrable for* $\mathbf{m}_E$ and

$$f(E)=\int f(\lambda)\,d\,\mathbf{m}_E(\lambda).$$

Here $\int f(\lambda)\,d\,\mathbf{m}_E(\lambda)$ is the *integral of f with respect to* $\mathbf{m}_E$. Thus if R is an unbounded scalar type spectral operator in G, then for all $f\in Bor(\sigma(R))$ such that $Dom(f(R))=\mathsf{D}$ *f is essentially integrable for* $\mathbf{m}_E$ and

$$f(R)=\int f(\lambda)\,d\,\mathbf{m}_E(\lambda).$$

5. Generalization of the Newton-Leibnitz formula

In this section we shall apply the results of the previous one in order to prove Newton-Leibnitz formulas for integration with respect to the $\sigma(B(G),\mathcal{N})-$topology, when $\mathcal{N}$ is an $E-$appropriate set with the duality property, for integration with respect to the sigma-weak topology, and for integration with respect to the weak operator topology.

COROLLARY 2.33 ($\sigma(B(G),\mathcal{N})-$ **Newton-Leibnitz formula 1**). *Let R be a possibly unbounded scalar type spectral operator in G, U an open neighborhood of $\sigma(R)$, $S : U \to \mathbb{C}$ an analytic map and $\mathcal{N}$ an $E-$appropriate set with the duality property. Assume that $S : U \to \mathbb{C}$ is an analytic map and there is $L > 0$ such that $]-L, L[\cdot U \subseteq U$ and*

(1) $\widetilde{S}_t \in \mathfrak{L}_E^{\infty}(\sigma(R))$, $\widetilde{\left(\frac{dS}{d\lambda}\right)_t} \in \mathfrak{L}_E^{\infty}(\sigma(R))$ *for all* $t \in]-L, L[$;

(2)

$$\int^{*} \left\| \widetilde{\left(\frac{dS}{d\lambda}\right)_t} \right\|_{\infty}^{E} dt < \infty$$

(here the upper integral is with respect to the Lebesgue measure on $]-L, L[$);

(3) *for all $\omega \in \mathcal{N}$ the map $]-L, L[\ni t \mapsto \omega\left(\frac{dS}{d\lambda}(tR)\right) \in \mathbb{C}$ is Lebesgue measurable.*

Then for all $u_1, u_2 \in]-L, L[$

$$R\int_{u_1}^{u_2} \frac{dS}{d\lambda}(tR)\,dt = S(u_2R) - S(u_1R) \in B(G).$$

Here the integral is the weak-integral of the map $[u_1, u_2] \ni t \mapsto \frac{dS}{d\lambda}(tR) \in B(G)$ with respect to the Lebesgue measure on $[u_1, u_2]$ and with respect to the $\sigma(B(G),\mathcal{N})-$ topology. Moreover if $\mathcal{N}$ is an $E-$appropriate set with the isometric duality property and $M \doteqdot \sup_{\sigma\in\mathcal{B}(\mathbb{C})} \|E(\sigma)\|_{B(G)}$ then

$$\left\| \int_{u_1}^{u_2} \frac{dS}{d\lambda}(tR)\,dt \right\|_{B(G)} \leq 4M \int_{[u_1,u_2]}^{*} \left\| \widetilde{\left(\frac{dS}{d\lambda}\right)_t} \right\|_{\infty}^{E} dt. \tag{164}$$

PROOF. Let μ the Lebesgue measure on $[u_1, u_2]$. By (71), the hypotheses, and statement (c) of Theorem $18.2.11$ of **[DS]** we have

a: $(\forall t \in [u_1, u_2])(S(tR) \in B(G))$;
b: $(\forall t \in [u_1, u_2])(\frac{dS}{d\lambda}(tR) \in B(G))$;
c: $([u_1, u_2] \ni t \mapsto \|\frac{dS}{d\lambda}(tR)\|_{B(G)}) \in \mathfrak{F}_1([u_1, u_2]; \mu)$,

So by hypothesis (3), by (c) and Theorem 2.2 we have that the map

$$[u_1, u_2] \ni t \mapsto \frac{dS}{d\lambda}(tR) \in \langle B(G), \sigma(B(G),\mathcal{N})\rangle \tag{165}$$

is scalarly essentially $(\mu, B(G))-$ integrable and if $\mathcal{N}$ is an $E-$appropriate set with the isometric duality property then its weak-integral satisfies (164). This means that, made exception for (149), all the hypotheses of Theorem 2.25 hold for $X \doteqdot [u_1, u_2]$, $h \doteqdot (S_{u_2} - S_{u_1}) \upharpoonright \sigma(R)$, $g : \sigma(R) \ni \lambda \mapsto \lambda \in \mathbb{C}$ and finally for the map $[u_1, u_2] \ni t \mapsto f_t \doteqdot \left(\frac{dS}{d\lambda}\right)_t \upharpoonright \sigma(R)$.

Next let $\sigma \in \mathcal{B}(\mathbb{C})$ be bounded, so by Key Lemma 1.7 $R_\sigma \upharpoonright G_\sigma$ is a scalar type spectral operator such that $R_\sigma \upharpoonright G_\sigma \in B(G_\sigma)$, moreover by (43) U is an open neighborhood of $\sigma(R_\sigma \upharpoonright G_\sigma)$. Thus we can apply statement (3) of Theorem 1.21 to the Banach space G_σ, the analytic map S and to the operator $R_\sigma \upharpoonright G_\sigma$. In particular the map $[u_1, u_2] \ni$

$t \mapsto \frac{dS}{d\lambda}(t(R_\sigma \upharpoonright G_\sigma)) \in B(G_\sigma)$ is Lebesgue integrable in $\|\cdot\|_{B(G_\sigma)}-$topology, that is in the meaning of Definition 2, $n°4$, §3, Ch. IV of [**INT**]. By Lemma 2.17 $\xi_\sigma \in B(B(G_\sigma), B(G))$, so

$$\mathcal{N}_\sigma \subset B(G_\sigma)^*.$$

Therefore we deduce by using Theorem 1, $IV.35$ of the [**INT**], that for all $\omega_\sigma \in \mathcal{N}_\sigma$ the map $[u_1, u_2] \ni t \mapsto \omega_\sigma\left(\frac{dS}{d\lambda}(t(R_\sigma \upharpoonright G_\sigma))\right) \in \mathbb{C})$ is Lebesgue integrable, in addition for all $\omega_\sigma \in \mathcal{N}_\sigma$

$$\int_{u_1}^{u_2} \omega_\sigma\left(\frac{dS}{d\lambda}(t(R_\sigma \upharpoonright G_\sigma))\right) dt = \omega_\sigma\left(\oint_{u_1}^{u_2} \frac{dS}{d\lambda}(t(R_\sigma \upharpoonright G_\sigma))\, dt\right).$$

Thus we can state that $[u_1, u_2] \ni t \mapsto \frac{dS}{d\lambda}(t(R_\sigma \upharpoonright G_\sigma)) \in \langle B(G_\sigma), \sigma(B(G_\sigma), \mathcal{N}_\sigma)\rangle$ is scalarly essentially $(\mu, B(G_\sigma))-$integrable and its weak-integral is such that

$$\int_{u_1}^{u_2} \frac{dS}{d\lambda}(t(R_\sigma \upharpoonright G_\sigma))\, dt = \oint_{u_1}^{u_2} \frac{dS}{d\lambda}(t(R_\sigma \upharpoonright G_\sigma))\, dt. \tag{166}$$

Moreover by statement (3) of Theorem 1.21

$$(R_\sigma \upharpoonright G_\sigma)\oint_{u_1}^{u_2} \frac{dS}{d\lambda}(t(R_\sigma \upharpoonright G_\sigma))\, dt = S(u_2(R_\sigma \upharpoonright G_\sigma)) - S(u_1(R_\sigma \upharpoonright G_\sigma)).$$

Thus by (166)

$$(R_\sigma \upharpoonright G_\sigma)\int_{u_1}^{u_2} \frac{dS}{d\lambda}(t(R_\sigma \upharpoonright G_\sigma))\, dt = S(u_2(R_\sigma \upharpoonright G_\sigma)) - S(u_1(R_\sigma \upharpoonright G_\sigma)). \tag{167}$$

This implies exactly the hypothesis (149) of Theorem 2.25, by choosing for example $\sigma_n \doteqdot B_n(\mathbf{0})$, for all $n \in \mathbb{N}$. Therefore by Theorem 2.25 we obtain the statement. $\square$

COROLLARY 2.34 ($\sigma(B(G), \mathcal{N})-$ **Newton-Leibnitz formula 2**). *Let R be a possibly unbounded scalar type spectral operator in G, U an open neighborhood of $\sigma(R)$, $S : U \to \mathbb{C}$ an analytic map and $\mathcal{N}$ an $E-$appropriate set with the duality property. Assume that there exists $L > 0$ such that $]-L, L[\cdot U \subseteq U$ and for all $t \in]-L, L[$, $\widetilde{S}_t \in \mathfrak{L}_E^\infty(\sigma(R))$ and there exists $K_0 \subset]-L, L[$ such that $\complement K_0$ is a Lebesgue negligible set and for all $t \in K_0$, $\widetilde{\left(\frac{dS}{d\lambda}\right)}_t \in \mathfrak{L}_E^\infty(\sigma(R))$ moreover*

(1) *there is $F :]-L, L[\to B(G)$ extending $K_0 \ni t \mapsto \frac{dS}{d\lambda}(tR) \in B(G)$ such that*

$$\int^* \|F(t)\|_{B(G)}\; dt < \infty$$

(here the upper integral is with respect to the Lebesgue measure on $]-L, L[$),

(2) *for all $\omega \in \mathcal{N}$ the map $]-L, L[\ni t \mapsto \omega(F(t)) \in \mathbb{C}$ is Lebesgue measurable.*

Then for all $u_1, u_2 \in]-L, L[$

$$R\int_{u_1}^{u_2} \frac{dS}{d\lambda}(tR)\, dt = S(u_2 R) - S(u_1 R) \in B(G).$$

Here the integral is the weak-integral of the map $[u_1, u_2] \ni t \mapsto \frac{dS}{d\lambda}(tR) \in B(G)$ with respect to the Lebesgue measure on $[u_1, u_2]$ and with respect to the $\sigma(B(G), \mathcal{N})-$ topology. Moreover if $\mathcal{N}$ is an $E-$appropriate set with the isometric duality property then

$$\left\|\int_{u_1}^{u_2} \frac{dS}{d\lambda}(tR)\, dt\right\|_{B(G)} \leq \int_{[u_1, u_2]}^* \left\|\frac{dS}{d\lambda}(tR)\right\|_{B(G)} dt.$$

PROOF. By Theorem 2.2 and (118)

$$[u_1, u_2] \ni t \mapsto \frac{d\,S}{d\,\lambda}(tR) \in \langle B(G), \sigma(B(G), \mathcal{N})\rangle$$

is scalarly essentially $(\mu, B(G))-$integrable and if $\mathcal{N}$ is an $E-$appropriate set with the isometric duality property its weak integral satisfies by (117) the inequality in the statement. Thus the proof goes on as that in Corollary 2.33. □

COROLLARY 2.35 (Sigma-Weak Newton-Leibnitz formula). *The statement of Corollary 2.33 (respectively Corollary 2.34) holds if G is a complex Hilbert space and everywhere $\mathcal{N}$ is replaced by $\mathcal{N}_{pd}(G)$.*

PROOF. By Remark 2.12, $\mathcal{N}_{pd}(G)$ is an $E-$appropriate set with the isometric duality property, hence the statement by Corollary 2.33 (respectively Corollary 2.34). □

COROLLARY 2.36 (Weak Newton-Leibnitz formula). *The statement of Corollary 2.33 (respectively Corollary 2.34) holds if G is a reflexive complex Banach space and everywhere $\mathcal{N}$ is replaced by $\mathcal{N}_{st}(G)$.*

PROOF. By using Corollary 2.6 instead of Theorem 2.2, we obtain (165) and (164) by replacing $\mathcal{N}$ with $\mathcal{N}_{st}(G)$. Then the proof procedes similarly to that of Corollary 2.33 (respectively Corollary 2.34). □

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